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Asymptotic-Preserving Schemes for Kinetic-Fluid Modeling of Mixture Flows

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Abstract. We consider coupled models for particulate flows, where the disperse phase is made of particles with distinct sizes. We are thus led to a system coupling the incompressible Navier–Stokes equations to the multi-component Vlasov–Fokker–Planck equations. We design an *asymptotic-preserving* numerical scheme to approximate the system. The scheme is based on suitable implicit treatment of the stiff drag force term as well as the Fokker–Planck operator, and can be formally shown to capture the hydrodynamic limit with time step and mesh size independent of the Stokes number. Numerical examples illustrate the accuracy and asymptotic behavior of the scheme, with several interesting applications.

AMS subject classifications: 35Q35, 65M06

Key words: Particulate flows, coupled kinetic-fluid model, Vlasov-Fokker-Planck-Navier-Stokes equations, asymptotic preserving schemes.

1 Introduction

This paper concerns the kinetic-fluid models for a mixture of flows in which the particles represent the disperse phase evolving in a dense fluid. Applications of such kinetic-fluid models include the dispersion of smoke or dust [14], biomedical modeling of spray [3], coupled models in combustion theory [35], etc. Specifically, we focus on the models that describe a large number of particles, *with distinct but fixed sizes,* interacting with a fluid. Here the dense fluid phase is modeled by the Euler or Navier-Stokes equations and particles dispersed in the fluid are modeled by Fokker-Planck type kinetic equations. Such

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multi-size particle systems have a wide range of applications in engineering, especially for the complex meteorological simulation of large aircraft icing process. For the distribution of droplets in the air, the influence of large droplets contained in the droplet distribution cannot be ignored [12,22,33]. However, it is very difficult to simulate multi-size particles by experimental means. See [20,34] for more details on the modeling of such multi-phase flows.

In this paper, we study the model where the evolution of the particle distribution function is driven by a combination of particle transport, a drag force exerted by the surrounding fluid on the particle obeying the Stokes Law, an external potential (such as gravity, electrostatic force, centrifugal force, etc.) and Brownian motion of particles. For the two physically important regimes first investigated by Goudon et al. [16,17], we focus on the fine particle regime given in [17]. The fluid phase is incompressible and viscous, all phases are isothermal, with interactions including coagulation and fragmentation that occur between particles, while the change of particle sizes is ignored. For the sake of simplicity, we suppose that the fluid density is constant and homogeneous. In this model, particles and fluid systems are coupled through nonlinear terms. Such a coupling and nonlinearities pose difficulties in mathematical analysis and numerical computations when compared with uncoupled problems. Furthermore, from a numerical point of view, the kinetic framework leads to high computational costs in both size and time, posing further computational challenges.

The study of existence, uniqueness and regularity problems depends on the nature of the coupling and the complexity of the equations used to describe the fluid. For the two-phase flow model system, it is worth mentioning related works like existence of strong solutions locally in time without velocity-diffusion [4], existence of weak solutions for the Vlasov–Stokes system [21] and for the incompressible Vlasov–Navier–Stokes system on a periodic domain [6] or a bounded domain [36], global-in-time existence of classical solutions close to the equilibrium for the incompressible Navier–Stokes–Vlasov–Fokker–Planck system [15], analysis of compressible models [29], several studies of coupling with the Euler system without viscosity [7,9] and systems with energy exchanges [5]. Analysis of the asymptotics in the two-phase flow system is due to [16,17] by means of relative entropy methods, see also [30]. For the multi-phase flow model system (2.1), existence has been discussed in [20] and regularity properties of the solutions close to the equilibrium as well as its long time behavior have been investigated recently by the authors in [26].

Numerical methods for such particulate flows have been developed in recent years, including particle-in-cell method [1], Eulerian–Lagrangian method [31,32], level set approach [28] and so on. One of the difficulties in numerically solving such multi-component Vlasov-Fokker-Planck-Navier-Stokes systems comes from the varying Stokes number ε, which describes the ratio of the Stokes settling time over a certain time unit of observation. Due to the multiscale nature of the problem, it is often desired to design numerical schemes that possess the asymptotic-preserving (AP) properties [8,18,19,27]. The AP schemes (first named in [23]) refer to those that, when letting the Stokes num-