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Local Virtual Element Basis Functions for Space-Time Discontinuous Galerkin Schemes on Unstructured Voronoi Meshes

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Abstract. We introduce a new class of Discontinuous Galerkin (DG) methods for solving nonlinear conservation laws on unstructured Voronoi meshes that use a local Virtual Element basis defined within each polygonal control volume. The basis functions are evaluated as an L_2 projection of the virtual basis which remains unknown, along the lines of the Virtual Element Method (VEM). Contrary to the VEM approach, the n ew basis functions lead to a nonconforming representation of the solution with dis-c ontinuous data across the element boundaries, as employed in DG discretizations. The discretization in time is carried out following the ADER (Arbitrary order DERivative Riemann problem) methodology, which yields one-step fully discrete schemes that make use of a coupled space-time representation of the numerical solution. The space-time basis functions are constructed as a tensor product of the novel local virtual basis in space and a one-dimensional Lagrange nodal basis in time. The resulting space-time stiffness matrix is stabilized by an extension of the dof–dof stabilization technique adopted in the VEM framework, hence allowing an element-local space-time Galerkin finite element predictor to be evaluated. The new VEM-DG algorithms are rigorously validated against a series of benchmarks in the context of compressible Euler and Navier–Stokes equations.

AMS subject classifications: 65Mxx, 65Yxx

Key words: Discontinuous Galerkin, Virtual Element Method, high order in space and time, ADER schemes, unstructured meshes.

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1 Introduction

Discontinuous Galerkin (DG) methods have been first introduced in [69] for the solution of neutron transport equations, and subsequently applied to general nonlinear systems of hyperbolic conservation laws in [33–36]. In the DG framework, the numerical solution is represented by piecewise polynomials within each mesh control volume, allowing jumps of the discrete solution across element boundaries. The main advantage of the DG approach is that it automatically provides high order of accuracy *locally*, thus it does not require any reconstruction stencil like in finite volume solvers. Furthermore, DG schemes are typically more accurate compared to finite volume or finite difference methods, because the entire high order polynomial is evolved in time for each computational cell.

The discrete solution is represented in terms of an expansion that involves a set of basis functions and the associated degrees of freedom, also referred to as expansion coefficients. The basis can be either modal or nodal, provided that it achieves the formal order of accuracy of the method. In the nodal approach, the degrees of freedom correspond to the value of the numerical solution at the nodal points, while in the modal approach the expansion coefficients give the modes of the polynomial basis.

Efficient DG schemes can be devised with *nodal basis* if the nodes are carefully chosen, hence leading to nodal DG schemes with Gauss-Lobatto nodes [45] or Gauss-Legendre nodes [70]. These schemes are typically referred to as spectral element methods (SEM) [60], which fit the Summation-By-Parts Simultaneous-Approximation-Term (SBP-SAT) framework [30, 52]. The nodal basis functions are defined on a reference element, to which the physical control volume is mapped. This means that the usage of nodal basis is restricted to either Cartesian meshes with quadrilaterals and hexahedra, or unstructured grids made of simplex control volumes, namely triangles and tetrahedra. To overcome this limitation, in [50] a nodal basis ansatz on more general unstructured meshes is proposed, and recently in [26] an agglomerated continuous finite element basis is devised at the sub-grid level of a Voronoi mesh. The nonlinear stability of nodal DG schemes is guaranteed by employing either artificial viscosity techniques [51, 67, 75, 80] or sub-cell finite volume limiters [44, 56].

Alternatively, *modal basis* like Taylor basis can be easily employed on very general control volumes [23, 46], although the associated computational cost increases since no reference element is available. Nevertheless, the adoption of a hierarchical modal approach is very convenient for designing slope and moment limiters in order to ensure the stability of DG schemes [1, 37, 41, 61, 68].

With the aim of dealing with general polygonal and polyhedral elements, the Virtual Element Method (VEM), which belongs to the class of continuous finite element methods, has recently emerged [2, 10, 11, 13, 14, 81]. In the VEM framework, the discrete solution is approximated by a set of basis functions that do not need to be explicitly determined, hence making them only *virtually* defined. Indeed, the numerical solution is known using suitable operators that project the basis functions onto polynomial spaces of any degree, allowing for the discretization and appropriate approximation of the continuous linear