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A Sparse Grid Wavelet Galerkin Method for 3-D Static Piezoelectric Equations

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Abstract. The 3-D static piezoelectric equations with four unknowns are fundamental in studying physical behavior of piezoelectric materials. It is very challenging to numerically solve the equations efficiently due to the so-called "curse of dimensionality". In this paper, we propose a sparse grid wavelet Galerkin method to overcome the difficulty, by combining the wavelet Galerkin method and the sparse grid technique. The error analysis is also developed in detail. Both the theoretical analysis and numerical results show the previous method outperforms the usual finite element method.

AMS subject classifications: 65D40, 65N30, 65T60, 65N12

Key words: Piezoelectric equations, sparse grid method, sparse tensor product method, wavelet Galerkin method.

1 Introduction

Piezoelectric materials are a kind of smart materials, which have applications in a variety of industrial and engineering devices, such as surface acoustic wave (SAW) devices [39], sonar devices [37] and aerospace equipments [54]. All of these applications involve the so-called positive and reverse piezoelectric effect (cf. [16, 17, 36]), that means, when a piezoelectric material is mechanically deformed, a voltage is generated inside the material, and conversely electric field can cause mechanical deformation of the piezoelectric material. These phenomenon can be described mathematically in terms of the piezoelectric equations—a coupled system of partial differential equations with respect to the displacement and the electric potential. There are very few piezoelectric problems having close-formed solutions (cf. [35, 41, 42, 49]), so it is very important to propose efficient numerical methods for such equations.

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The finite element method (FEM) is a typical way for solving the piezoelectric equations (cf. [1,2,7,32,33,55]). This method has a remarkable advantage in resolving physical problems over complicated geometric domains. However, in the three-dimensional (3-D) case, since there are four unknowns (three for the displacement and one for the electric potential), it is very difficult to store and solve the resulting very large scale linear system discretized from FEMs. The boundary element method (BEM) is another typical approach to solving the piezoelectric problems (cf. [20,21,28,34,48,52]). It uses the dual reciprocity method to transform the original problem to the integral equations on the boundary, reducing the dimension of the solution domain. However, one needs to borrow some very subtle techniques to solve the discrete problem. In the past decade, some meshless methods are also devised to solve the piezoelectric equations (cf. [4,6,40,53,56,57]). Generally speaking, the meshless method uses a series of appropriate scattered nodes to express the problem domain and boundary, and constructs basis functions based on these nodes. This method has certain advantage in handling problems with complex geometries, and can conveniently improve the accuracy of the numerical solution locally [40].

For the 3-D piezoelectric equations, because of the "curse of dimensionality", the efficiency of all the above methods are deteriorated significantly, which means that the increase in the number of degrees of freedom (DOFs) leads to a much lower convergence rate due to the high dimensionality of the problem. Furthermore, there are few mathematical studies on numerical methods for piezoelectric equations, most existing results working on convergence analysis of the numerical methods through numerical experiments. Therefore, in this paper we are going to propose an efficient method for the piezoelectric equations over regular domains and then develop the convergence analysis for the method proposed as well.

The sparse grid method, also known as the sparse tensor product method, is an effective way to overcome the "curse of dimensionality", if the dimension of independent variables is not very large (cf. [22–25, 51]). The method's concept can be dated back to Smoljaks' construction of multivariate quadrature formulas (cf. [45]). Its main ideas consist in carrying out a multi-scale decomposition of a vector space to create wavelet spaces, and then forming a sparse tensor product space by means of only a small number of wavelet bases in each space component. The method has a remarkable advantage, that is, it often converges at a rate similar to the full space when the approximate function satisfies the required regularity. The sparse grid method has been applied to resolve several mathematical physics problems. For example, it was combined with the discontinuous Galerkin method to solve higher dimensional elliptic equations (cf. [50]) and was also applied to radiative transfer equations (cf. [30]). Some other applications can be found in [26,31,43,58]. All these works have demonstrated the method's efficiency in simulating several high dimensional problems. The construction of a sparse grid method has a close relation with the use of wavelets. We mention that the wavelet method itself is a well-developed numerical method (cf. [14, 15]). For example, the wavelet Galerkin method has been applied to some classical partial differential equations, such as the Helmholtz equation [44] and the Stokes equation [19].