Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0258

Vol. **36**, No. 2, pp. 551-580
August 2024

# A Parallel Domain Decomposition Method for the Fully-Mixed Stokes-Dual-Permeability Fluid Flow Model with Beavers-Joseph Interface Conditions

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Received 1 October 2023; Accepted (in revised version) 29 May 2024

**Abstract.** In this paper, a parallel domain decomposition method is proposed for solving the fully-mixed Stokes-dual-permeability fluid flow model with Beavers-Joseph (BJ) interface conditions. Three Robin-type boundary conditions and a modified weak formulation are constructed to completely decouple the original problem, not only for the free flow and dual-permeability regions but also for the matrix and microfracture flow fields in the dual-porosity media. We derive the equivalence between the original problem and the decoupled systems with some suitable compatibility conditions, and also demonstrate the equivalence of two weak formulations in different Sobolev spaces. Based on the completely decoupled modified weak formulation, the mesh-independent geometric convergence rate of the proposed iterative parallel algorithm is proved rigorously with some suitable parameters. To carry out the convergence analysis of our proposed algorithm, we utilize an important but general convergence lemma for the steady-state problems. Finding the optimized Robin parameters that can accelerate the convergence of the proposed algorithm is an open problem mentioned inhere. Finally, several numerical experiments are presented to illustrate and validate the exclusive features of our proposed algorithm.

**AMS subject classifications**: 65M55, 65M60

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**Key words**: Stokes-dual-permeability model, Beavers-Joseph interface conditions, Robin-type domain decomposition, parallel computation, optimized Robin parameters.

## 1 Introduction

Multi-domain, multi-physics coupled problems are significant in many natural and industrial applications, such as the groundwater fluid flow in the karst aquifer, petroleum extraction, industrial filtration, blood flow motion in the arteries, and so on. Up to now, a great deal of mathematical and physical models for free flow coupled with complicated porous media have been constructed, including the Stokes-Darcy model [1–4], dual-porosity-Stokes model [5–11], etc. Typically, we usually utilize the Darcy equation in the traditional Stokes-Darcy fluid flow system to simulate the single porosity model. Plenty of numerical methods for solving the Stokes-Darcy system can be referred to Lagrange multiplier methods [12–14], domain decomposition methods [15–22], optimized Schwarz methods [23–25], to name just a few.

The Darcy equation in the Stokes-Darcy system has some limitations in describing fluid flow in porous media with complicated geometrical structures. For example, a naturally fractured reservoir containing the multi-porosity/permeability regions is comprised of low permeable rock matrix blocks surrounded by an irregular network of natural microfractures [5,6,26,27]. To describe the coupled flow in the dual-porosity media and conduits, Hou et al. [5] constructed a new dual-porosity-Stokes model, and proposed four physically valid interface conditions to couple the Stokes with the dual-porosity, including a no-exchange condition, a mass balance condition, a force balance condition, and the Beavers–Joseph (BJ) conditions. Their work well established the continuous model of the coupled dual-porosity-Stokes system to simulate multistage hydraulic fractured horizontal wellbore.

For the dual-porosity-Stokes system, one important task is how to develop effective numerical algorithms. Inspired by the decoupled ideas for the Stokes–Darcy model, some researchers extended to study the Stokes-dual-permeability model. A natural decoupled method is the domain decomposition method (DDM), since it can decouple the multi-domain, multi-physics problems naturally under the introduced interface boundary conditions [15–22], meanwhile, there are many well studied off-the-shelf and efficient solvers for each decoupled subproblem. Based on the characteristics of easy-to-operate, high efficiency and convenient parallel computing, DDM has received extensive attention and applications undoubtedly. In [10], the authors extended the Robin-Robin DDM of the Stokes-Darcy model in [16,18] to deal with the dual-porosity-Stokes model. Noting that such methods were usually based on Galerkin approximation to two elliptic pressure equations in the dual-porosity media. Hou et al. [10] studied the dual-porosity-conduit system with the Beavers-Joseph-Saffman (BJS) interface boundary conditions, decoupled the Stokes equations and the dual-porosity model by DDM, but not for matrix and microfracture systems. They demonstrated the convergence of the proposed algorithms