Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0011

Vol. **36**, No. 3, pp. 673-710
September 2024

Fast Implementation of FEM for Integral Fractional Laplacian on Rectangular Meshes

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Received 9 January 2023; Accepted (in revised version) 9 May 2023

Abstract. We show that the entries of the stiffness matrix, associated with the C^0-piecewise linear finite element discretization of the hyper-singular integral fractional Laplacian (IFL) on rectangular meshes, can be simply expressed as one-dimensional integrals on a finite interval. Particularly, the FEM stiffness matrix on uniform meshes has a block-Toeplitz structure, so the matrix-vector multiplication can be implemented by FFT efficiently. The analytic integral representations not only allow for accurate evaluation of the entries, but also facilitate the study of some intrinsic properties of the stiffness matrix. For instance, we can obtain the asymptotic decay rate of the entries, so the "dense" stiffness matrix turns out to be "sparse" with an $\mathcal{O}(h^3)$ cutoff. We provide ample numerical examples of PDEs involving the IFL on rectangular or L-shaped domains to demonstrate the optimal convergence and efficiency of this semi-analytical approach. With this, we can also offer some benchmarks for the FEM on general meshes implemented by other means (e.g., for accuracy check and comparison when triangulation reduces to rectangular meshes).

AMS subject classifications: 15B05, 41A05, 41A25, 74S05

Key words: Integral fractional Laplacian, nonlocal/singular operators, FEM on rectangular meshes, stiffness matrix with Toeplitz structure.

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1 Introduction

There has been a fast growing interest in nonlocal models in terms of numerics, analysis and applications, which can be testified by the recent review articles [8,20,31] and monographs [19,21], together with many references therein. Among several types of nonlocal operators, the integral fractional Laplacian (IFL) is deemed as one of the most prominent, but challenging operators to deal with. It is known that given a sufficiently nice function $u(\boldsymbol{x})\!:\!\mathbb{R}^d\!\to\!\mathbb{R}$ and $s\!\in\!(0,1)$, its IFL $(-\Delta)^s u(\boldsymbol{x})$ has the hypersingular integral representation (cf. [35]):

$$(-\Delta)^s u(\boldsymbol{x}) = C_{d,s}\,\text{p.v.}\int_{\mathbb{R}^d} \frac{u(\boldsymbol{x})-u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2s}}\,\mathrm{d}\boldsymbol{y}, \quad C_{d,s} := \frac{2^{2s} s\Gamma(s+d/2)}{\pi^{d/2}\Gamma(1-s)}, \tag{1.1}$$

where "p.v." stands for the principal value and $C_{d,s}$ is the normalisation constant. It can also be defined as a pseudo-differential operator with symbol $|\boldsymbol{\xi}|^{2s}$ through the Fourier transform:

$$(-\Delta)^s u(\boldsymbol{x}) = \mathscr{F}^{-1}\big[|\boldsymbol{\xi}|^{2s}\mathscr{F}[u](\boldsymbol{\xi})\big](\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^d. \tag{1.2}$$

The nonlocal and singular nature of this operator poses major difficulties in discretisation and analysis.

Most recent concerns are with PDEs involving the IFL operator on an open bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$. More precisely, given $f:\Omega\to\mathbb{R}$ in a suitable space, we look for u on Ω satisfying the fractional Poisson equation with the (nonlocal) homogeneous Dirichlet boundary condition:

$$(-\Delta)^s u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } \Omega; \quad u(\boldsymbol{x}) = 0 \quad \text{on } \Omega^c := \mathbb{R}^d\backslash\Omega. \tag{1.3}$$

We also intend to apply the FEM solver to spatial discretisation of the fractional diffusion equation:

$$u_t(\boldsymbol{x},t) + (-\Delta)^s u(\boldsymbol{x},t) = F(u(\boldsymbol{x},t)) \quad \text{in } \Omega\times(0,T], \tag{1.4}$$

with the boundary condition: $u=0$ in $\Omega^c\times[0,T]$ and the initial condition: $u|_{t=0}=u_0$ on $\bar{\Omega}$. Here, $F(u)$ is a certain nonlinear functional of u.

1.1 Contributions

In this paper, we provide a semi-analytic means for computing the piecewise linear FEM stiffness matrix for (1.3) and (1.4) on a rectangular domain Ω with a rectangular partition, or a more general domain that can be decomposed into occluded rectangular meshes, e.g., an L-shaped domain. More specifically, given a uniform partition (with mesh size h_x,h_y along x,y, respectively) of Ω with the C^0-piecewise linear tensorial FEM nodal basis: $\{\Phi_{mn}(\boldsymbol{x}) = \phi_m(x)\phi_n(y)\}_{1\le m\le M}^{1\le n\le N}$, the fractional stiffness matrix $\boldsymbol{S}=(S_{ll'})$ of size M^2N^2 is a block-Toeplitz matrix that can be generated by an $M\times N$ matrix $\boldsymbol{G}=(G_{kj})$ and each G_{kj}