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High Moment and Pathwise Error Estimates for Fully Discrete Mixed Finite Element Approximations of Stochastic Navier-Stokes Equations with Additive Noise

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Abstract. This paper is concerned with high moment and pathwise error estimates for fully discrete mixed finite element approximations of stochastic Navier-Stokes equations with general additive noise. The implicit Euler-Maruyama scheme and standard mixed finite element methods are employed respectively for the time and space discretizations. High moment error estimates for both velocity and time-averaged pressure approximations in strong L^2 and energy norms are obtained, pathwise error estimates are derived by using the Kolmogorov Theorem. Unlike their deterministic counterparts, the spatial error constants grow in the order of $\mathcal{O}(k^{-\frac{1}{2}})$, where k denotes time step size. Numerical experiments are also provided to validate the error estimates and their sharpness.

AMS subject classifications: 65N12, 65N15, 65N30

Key words: Stochastic Navier-Stokes equations, additive noise, Wiener process, Itô stochastic integral, mixed finite element methods, inf-sup condition, high moment, and pathwise error estimates.

1 Introduction

We consider the following time-dependent stochastic Navier-Stokes equations:

$$d\mathbf{u} = \left[\nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \mathbf{f}\right]dt + \mathbf{g}(t)dW(t) \qquad \text{a.s. in } D_T, \tag{1.1a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{a.s. in } D_T, \tag{1.1b}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{a.s. in } D, \tag{1.1c}$$

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where $D=(0,L)^2\subset\mathbb{R}^2$ represents a period of the periodic domain in $\mathbb{R}^2$, $\mathbf{u}$ and p stand for respectively the velocity field and the pressure of the fluid, $\{W(t);t\geq 0\}$ denotes real-valued Wiener process and $\mathbf{f}$ is a body force function. In addition, $\mathbf{g}$ is the diffusion coefficient (see Section 2.3 for its precise definition). Here we seek periodic-in-space solutions $(\mathbf{u},p)$ with period L, that is, $\mathbf{u}(t,\mathbf{x}+L\mathbf{e}_i)=\mathbf{u}(t,\mathbf{x})$ and $p(t,\mathbf{x}+L\mathbf{e}_i)=p(t,\mathbf{x})$ almost surely and for any $(t,\mathbf{x})\in(0,T)\times\mathbb{R}^d$ and $1\leq i\leq d$, where $\{\mathbf{e_i}\}_{\mathbf{i=1}}^{\mathbf{d}}$ denotes the canonical basis of $\mathbb{R}^d$.

Numerical analysis of (1.1) has been studied by several researchers. In [6] the authors established stability and convergence of the standard mixed finite element method of (1.1). Later in [8] the authors proved the rates of convergence in probability for the velocity approximation in the case of multiplicative noise. The main difficulty for establishing a strong convergence for any numerical approximation of (1.1) is the interplay between the nonlinearity and stochasticity of the equations. To compute or estimate quantities of stochastic interests such as the expectation and moments, all the norms must have another layer of integration which is the main reason why the classical Gronwall inequality argument fails. To overcome this difficulty, in [8] the authors introduced a sequence of sub-sample spaces that converge to the sample space under the probability measure. The error estimates on these sub-sample spaces are computed with partial expectations and are considered as weak convergence, and some improved error estimates of the same type were recently obtained in [18]. In [1,2] the authors showed the strong L^2-convergence of the mixed finite element method for (1.1) by estimating the error estimates on the complements of these sub-sample spaces, which then leads to the strong convergence with a logarithmic rate. Moreover, the authors were able to establish in [3] strong convergence with a polynomial rate in the case of a divergence-free additive noise. We note that all the above-mentioned error estimates for (1.1) are second-moment estimates and most are only for the velocity approximation. No high moment and pathwise error estimates have been reported in the literature so far. These missing error estimates are important to know because they provide different quantities of stochastic interests in practice.

The primary goals of this paper are to fill such a void in the case of general additive noise and to develop the analysis techniques for deriving high moment and pathwise error estimates for numerical nonlinear stochastic PDEs in general. It should be noted that the desired high moment and pathwise error estimates will be obtained for both the velocity and pressure approximations of the fully discrete mixed finite element method for (1.1). Our main ideas are to obtain the former based on an exponential stability estimate, which is inspired by a similar idea first introduced in [3], and a bootstrap technique, and to obtain the latter by using the Kolmogorov Theorem (see Theorem 2.1).

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries including the definition of variational solutions to (1.1) and the assumptions on the diffusion function $\mathbf{g}$. In Section 3, we introduce the time discretization for (1.1) in Algorithm 1 and establish some stability estimates for its solution, including an exponential stability estimate which plays a crucial role in our error analysis for the velocity approximation in Theorems 3.1, and 3.2, and the error estimates for the pressure