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A Comparative Study of Two Allen-Cahn Models for Immiscible N-Phase Flows by Using a Consistent and Conservative Lattice Boltzmann Method

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Abstract. In this work, we conduct a detailed comparison between two second-order conservative Allen-Cahn (AC) models [*Model A*: Zheng *et al.*, Phys. Rev. E 101, 0433202 (2020) and *Model B*: Mirjalili and Mani, J. Comput. Phys. 498, 112657 (2024)] for the immiscible N-phase flows. Mathematically, these two AC equations can be proved to be equivalent under some approximate conditions. However, the effects of these approximations are unclear from the theoretical point of view, and would be considered numerically. To this end, we propose a consistent and conservative lattice Boltzmann method for the AC models for N-phase flows, and present some numerical comparisons of accuracy and stability between these two AC models. The results show that both two AC models have good performances in accuracy, but the *Model B* is more stable for the realistic complex N-phase flows, although there is an adjustable parameter in the *Model A*.

AMS subject classifications: 76T30,76D05,76B45

Key words: Model comparisons, Allen-Cahn models, N-phase flows, lattice Boltzmann method.

1 Introduction

The phase-field method with an order parameter introduced to distinguish different phases, has been widely used in the study of multiphase flows [1–4] for its advantages

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in needless of explicit interface-tracking and volume conservation. In this method, the interface of different phases is considered to have a small but finite thickness, and the physical variables are smoothly changed across the interface. In the phase-field method, there are two popular models, i.e., the fourth-order Cahn-Hilliard (CH) equation [5] and the second-order conservative Allen-Cahn (AC) equation [6,7], that are usually adopted. However, these models cannot be directly used to study some more complex problems in nature and engineering processes, where more than two phases are included, for instance, the enhanced oil recovery [8,9], emulsion formation [10,11], and additive manufacturing [12]. Therefore, to extend the scope of application of the phase-field model to depict more realistic multiphase systems, the phase-field models for two-phase problems should be extended in some way for the N-phase flows ($N \geq 3$), and simultaneously, the extended models should follow the consistency of reduction, i.e., they can reduce to the models for $N-M$ phase flows when M phase ($M < N$) are absent in the system with N phases.

The CH equation is popular in the simulation of multiphase flows due to its properties in the volume conservation and thermodynamic consistency. Boyer *et al.* [4,13] elaborately designed a consistent bulk free energy for the ternary CH model, which can reduce to the double-well structure for two-phase cases when one phase disappears. Combing the distinct features of the available models [3,4], Kim [14] proposed a ternary CH model with the continuous surface tension formulation to account for the surface tension effect, which allows the extension to more than three components. Recently, Dong [15] presented a reduction-consistent and thermodynamically-consistent formulation for an isothermal mixture consisting of N immiscible incompressible fluids, where the free energy density function is equivalent to the form originally suggested in Ref. [16]. However, it is usually difficult or even impossible to maintain the boundedness of the order parameter in the fourth-order CH equation, and the shrinkage/expansion occurs and the volume of a specific phase enclosed by the interface would diffuse into another phase to restore the equilibrium hyperbolic tangent profile [17]. In this case, although the total mass of CH system is conserved, there is a mass leakage among different phases, which would effect the accuracy of the interface position [17,18].

On the other hand, the second-order conservative AC model is much simpler than CH equation, and has been another strategy to capture the phase interfaces in N-phase flows. Abadi *et al.* [19] extended the conservative AC equation to the ternary fluids system, and the corresponding model is symmetric with respect to the phases and reduction-consistent. However, the consistency of reduction can not be guaranteed for the multiphase system with more than three phases. Motivated by the work of Ref. [20], Zheng *et al.* [21] developed a reduction-consistent and conservative AC equation, where a modified Lagrange multiplier is adopted to satisfy the consistency of reduction for N-phase flows. More recently, Mirjalili and Mani [22] proposed an N-phase extension of second-order conservative AC equation [6], which is in a simple conservative form and satisfies the volume conservation, the consistency of reduction, the symmetry with respect to the phases, and the consistency of mass-momentum transport.