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Distributional Sensitivity for Uncertainty Quantification

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Abstract. In this work we consider a general notion of *distributional sensitivity*, which measures the variation in solutions of a given physical/mathematical system with respect to the variation of probability distribution of the inputs. This is distinctively different from the classical sensitivity analysis, which studies the changes of solutions with respect to the values of the inputs. The general idea is measurement of sensitivity of outputs with respect to probability distributions, which is a well-studied concept in related disciplines. We adapt these ideas to present a quantitative framework in the context of uncertainty quantification for measuring such a kind of sensitivity and a set of efficient algorithms to approximate the distributional sensitivity numerically. A remarkable feature of the algorithms is that they do not incur additional computational effort in addition to a one-time stochastic solver. Therefore, an accurate stochastic computation with respect to a prior input distribution is needed only once, and the ensuing distributional sensitivity computation for different input distributions is a post-processing step. We prove that an accurate numerical model leads to accurate calculations of this sensitivity, which applies not just to slowly-converging Monte-Carlo estimates, but also to exponentially convergent spectral approximations. We provide computational examples to demonstrate the ease of applicability and verify the convergence claims.

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1 Introduction

Uncertainty quantification (UQ) has become an important tool for modelling in recent years. Many physical systems have uncertainties caused by unknown parameters in the

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model or by measurement noise plaguing experiments. In such cases, it is critical to understand and predict how the uncertainty affects quantities of interest (QoI) of the systems. This introduces a new paradigm for scientific computing and extends the traditional deterministic simulations to stochastic simulations.

One of the major challenges for stochastic computation and UQ is the simulation cost, as the dimensionality of simulations depends on the total number of random variables that one employs to parameterize the inputs. The larger the dimensionality the higher the cost, (the curse of dimensionality). To circumvent the difficulty, it is crucial to conduct *sensitivity analysis* (SA) prior to simulations. The goal of the sensitivity analysis is to determine which input variables have notable effects on the QoI and eliminate those with negligible effects on the simulation.

In this work we discuss a different kind of sensitivity analysis-distributional sensitivity analysis (DSA), which is intended to quantify the impact on the QoI with respect to changes in the probability distribution of the inputs. This is motivated by the fact that in many cases there is not sufficient data or evidence to fully specify the probability distribution of the inputs. Such kind of uncertainty is often referred to as *epistemic uncertainty*, as opposed to *aleatory uncertainty* where probabilistic information about the inputs is fully specified. For many practical systems, uncertain inputs often present themselves in the form of epistemic uncertainty, and acquiring more information to specify their probability can be a (highly) costly, and sometimes impossible, task. One of the immediate goals of DSA is to provide a guideline to direct the modeling effort. For inputs with large distributional sensitivity (DS), more effort will be required to acquire their probabilistic information; for inputs with small and negligible DS, it is acceptable to specify their distribution with something of computational convenience. By doing so, we can reduce the total number of epistemic variables to a minimum. It is worth remarking on the difference between the DS and the traditional sensitivity. While an input with small sensitivity in the traditional sense naturally implies small DS, there is no direct association on the other hand, i.e., an input with large sensitivity in the traditional sense does not necessarily imply large DS, and vice versa. Therefore, while the traditional SA is a necessary step to reduce the computational burden for (aleatory) stochastic simulations, the DSA is a necessary step to reduce the simulation effort for dealing with epistemic uncertainty.

Indeed, the underlying concern of DSA is the study of how assumptions about probability densities affects outputs. Unsurprisingly, this notion already exists in related fields; one manifestation of this is the "score function" approach [1, 13]. The score function method assumes a parameterization of a family of input distributions and primarily uses a Monte-Carlo estimate to compute sensitivities. The "what-if" problem, extrapolation of the QoI values to unsimulated density locations, is not a consideration of this paper. The spirit of DSA is also captured in the study of local/global sensitivity analysis from Bayesian statistics; this analysis studies the effect that the assumed prior has on the resulting posterior [4, 9, 14]. Our problem is not in the context of Bayesian statistics; in particular we are not concerned with updating our assumed "prior".

Since computational effort is of great concern in stochastic computations, it is desir-