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A Tailored Finite Point Method for Solving Steady MHD Duct Flow Problems with Boundary Layers

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Abstract. In this paper we propose a development of the finite difference method, called the tailored finite point method, for solving steady magnetohydrodynamic (MHD) duct flow problems with a high Hartmann number. When the Hartmann number is large, the MHD duct flow is convection-dominated and thus its solution may exhibit localized phenomena such as the boundary layer. Most conventional numerical methods can not efficiently solve the layer problem because they are lacking in either stability or accuracy. However, the proposed tailored finite point method is capable of resolving high gradients near the layer regions without refining the mesh. Firstly, we devise the tailored finite point method for the scalar inhomogeneous convection-diffusion problem, and then extend it to the MHD duct flow which consists of a coupled system of convection-diffusion equations. For each interior grid point of a given rectangular mesh, we construct a finite-point difference operator at that point with some nearby grid points, where the coefficients of the difference operator are tailored to some particular properties of the problem. Numerical examples are provided to show the high performance of the proposed method.

AMS subject classifications: 65N06, 65N12, 76W05

Key words: Magnetohydrodynamic equations, Hartmann numbers, convection-dominated problems, boundary layers, tailored finite point methods, finite difference methods.

1 Introduction

The purpose of this paper is to devise an efficient tailored finite point method for approximating the solution of magnetohydrodynamic (henceforth, MHD) duct flow problems at

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high Hartmann numbers. We study the problem of finding the velocity u and the induced magnetic field b for a laminar, fully developed flow of an incompressible, viscous, electrically conducting fluid in a straight channel. The channel has a uniform cross-section Ω which is an open bounded region in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. The fluid is driven by a constant mechanical pressure gradient $-dp/dz$. The direction of the constant transverse external magnetic field b_0 may be arbitrary to the x-axis, and both the velocity u and the induced magnetic field b are parallel to the z-axis.

The generalized equations of the MHD duct flow described above with suitable boundary conditions can be posed in dimensionless form as follows [14,16]:

$$\begin{cases} -\varepsilon\Delta u+\boldsymbol{a}\cdot\nabla b=f_1, & \text{in } \Omega, \\ -\varepsilon\Delta b+\boldsymbol{a}\cdot\nabla u=f_2, & \text{in } \Omega, \\ u=g_1, & \text{on } \partial\Omega, \\ b=g_2, & \text{on } \Gamma_D, \\ \nabla b\cdot\boldsymbol{n}=g_3, & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $u=u(x,y)$ and $b=b(x,y)$ are the velocity and the induced magnetic field in the z-direction, respectively; ε is the diffusivity coefficient with $0<\varepsilon:=1/Ha<1$ and $Ha:=b_0 l\sqrt{\delta/\mu}$ is the Hartmann number, b_0 is the intensity of the external magnetic field, l is the characteristic length of the duct, δ and μ are the electric conductivity and coefficient of viscosity of the fluid respectively; the convection field is given by

$$\boldsymbol{a}:=(a_1,a_2)^\top=(-\sin\alpha,-\cos\alpha)^\top,$$

$\alpha\in[0,\pi/2]$ is the angle from the positive y-axis to the externally applied magnetic field b_0, measured in the clockwise direction; f_i for $i=1,2$ are the given source terms and in most practical MHD applications, we have

$$f_1\equiv\varepsilon \quad \text{or} \quad f_1\equiv 0 \quad \text{and} \quad f_2\equiv 0, \quad \text{in} \quad \Omega,$$

g_i for $i=1,2,3$ are prescribed boundary data;

$$\partial\Omega=\Gamma_D\cup\Gamma_N \quad \text{with} \quad \Gamma_D\cap\Gamma_N=\varnothing \quad \text{and} \quad |\Gamma_D|>0;$$

$\boldsymbol{n}$ is the outward unit normal vector to Γ_N.

There are many investigations which use various numerical methods such as finite difference, finite element and boundary element methods to solve the MHD duct flow problems. We refer the reader to [8,12–14,16–18] and many references cited therein. However, when the Hartmann number Ha is large, such an MHD duct flow problem consists of a coupled system of convection-dominated convection-diffusion equations. It is well known that the solution of convection-dominated problems may exhibit localized phenomena such as boundary or interior layers, i.e., narrow regions where some derivative of solution rapidly changes. Most conventional numerical methods can not efficiently