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All Speed Scheme for the Low Mach Number Limit of the Isentropic Euler Equations

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Abstract. An all speed scheme for the Isentropic Euler equations is presented in this paper. When the Mach number tends to zero, the compressible Euler equations converge to their incompressible counterpart, in which the density becomes a constant. Increasing approximation errors and severe stability constraints are the main difficulty in the low Mach regime. The key idea of our all speed scheme is the special semi-implicit time discretization, in which the low Mach number stiff term is divided into two parts, one being treated explicitly and the other one implicitly. Moreover, the flux of the density equation is also treated implicitly and an elliptic type equation is derived to obtain the density. In this way, the correct limit can be captured without requesting the mesh size and time step to be smaller than the Mach number. Compared with previous semi-implicit methods [11,13,29], firstly, nonphysical oscillations can be suppressed by choosing proper parameter, besides, only a linear elliptic equation needs to be solved implicitly which reduces much computational cost. We develop this semi-implicit time discretization in the framework of a first order Local Lax-Friedrichs (or Rusanov) scheme and numerical tests are displayed to demonstrate its performances.

AMS subject classifications: 65M06, 65Z05, 76N99, 76L05

Key words: Low Mach number, Isentropic Euler equations, compressible flow, incompressible limit, asymptotic preserving, Rusanov scheme.

1 Introduction

Singular limit problems in fluid mechanics have drawn great attentions in the past years, like low-Mach number flows, magneto-hydrodynamics at small Mach and Alfven numbers and multiple-scale atmospheric flows. As mentioned in [17], the singular limit regime induces severe stiffness and stability problems for standard computational techniques. In this paper, we focus on the simplest Isentropic Euler equations and propose a

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numerical scheme that is uniformly applicable and efficient for all ranges of Mach numbers.

The problem under study is the Isentropic Euler equations

$$\begin{cases} \partial_t \rho_\epsilon + \nabla\cdot(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \\ \partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \nabla(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \dfrac{1}{\epsilon^2}\nabla p_\epsilon = 0, \end{cases} \tag{1.1}$$

where ρ_ϵ,$\rho_\epsilon \mathbf{u}_\epsilon$ is the density and momentum of the fluid respectively and ϵ is the scaled Mach number. This is one of the most studied nonlinear hyperbolic systems. For standard applications, the equation of state takes the form

$$p(\rho) = \Lambda \rho^\gamma, \tag{1.2}$$

where Λ,γ are constants depending on the physical problem.

It is rigorously proved by Klainerman and Majda [15,16] that when $\epsilon \to 0$, i.e., when the fluid velocity is small compared with the speed of sound [3], the solution of (1.1) converges to its incompressible counterpart. Formally, this can be obtained by inserting the expansion

$$\rho_\epsilon = \rho_0 + \epsilon^2 \rho_{(2)} + \cdots, \tag{1.3a}$$

$$\mathbf{u}_\epsilon = \mathbf{u}_0 + \epsilon^2 \mathbf{u}_{(2)} + \cdots, \tag{1.3b}$$

into (1.1) and equate the same order of ϵ. The limit reads as follows [15,18]:

$$\rho = \rho_0, \tag{1.4a}$$

$$\nabla\cdot \mathbf{u}_0 = 0, \tag{1.4b}$$

$$\partial_t \mathbf{u}_0 + \nabla(\mathbf{u}_0 \otimes \mathbf{u}_0) + \nabla p_{(2)} = 0. \tag{1.4c}$$

Here $p_{(2)}$ is a scalar pressure which can be viewed as the Lagrange multiplier of the incompressibility constraint. In view of the discussion of [17,18], p_0 is the thermodynamic pressure, which is uniform in the low Mach number limit, and $p_{(2)}$ is the hydrodynamic pressure. Low Mach number flows are flows which are slow compared with the speed of sound. In such a situation, pressure waves become very fast and, in the zero Mach number limit, an instantaneous pressure equalization takes place [24,25].

For atmosphere-ocean computing or fluid flows in engineering devices, when ϵ is small in (1.1), standard numerical methods become unacceptably expensive. Indeed, (1.1) has wave speeds of the form

$$\lambda = \mathbf{u}_\epsilon \pm \frac{1}{\epsilon}\sqrt{p'(\rho_\epsilon)},$$

where $p'(\rho_\epsilon)$ is the derivative with respect to ρ_ϵ. If a standard hyperbolic solver is used, the CFL requirement is $\Delta t = \mathcal{O}(\epsilon \Delta x)$. Moreover in order to maintain stability, the numerical dissipation required by the hyperbolic solver is proportional to $|\lambda|$. If $|\lambda| = \mathcal{O}(1/\epsilon)$,