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REVIEW ARTICLE

Velocity-Based Moving Mesh Methods for Nonlinear Partial Differential Equations

M. J. Baines[1,*], M. E. Hubbard[2] and P. K. Jimack[2]

[1] *Department of Mathematics, The University of Reading, RG6 6AX, UK.*
[2] *School of Computing, University of Leeds, LS2 9JT, UK.*

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Abstract. This article describes a number of velocity-based moving mesh numerical methods for multidimensional nonlinear time-dependent partial differential equations (PDEs). It consists of a short historical review followed by a detailed description of a recently developed multidimensional moving mesh finite element method based on conservation. Finite element algorithms are derived for both mass-conserving and non mass-conserving problems, and results shown for a number of multidimensional nonlinear test problems, including the second order porous medium equation and the fourth order thin film equation as well as a two-phase problem. Further applications and extensions are referenced.

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*Corresponding author. *Email addresses:* `m.j.baines@reading.ac.uk` (M. Baines), `m.e.hubbard@leeds.ac.uk` (M. E. Hubbard), `p.k.jimack@leeds.ac.uk` (P. K. Jimack)

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1 Introduction

This paper reviews some velocity-based moving mesh numerical methods for nonlinear time-dependent partial differential equations (PDEs). Numerous physical and biological applications of mathematics are governed by these PDEs which often exhibit complex behaviour that is difficult to predict in advance. For example, problems may be posed on moving domains which are determined implicitly by the solution of the equations.

Many PDE problems possess analytic properties, for example the invariance of certain integrals or the existence of solutions with a specific structure. The construction of numerical approximations which preserve such properties is one of the aims of geometric integration [28–30]. Where such *a priori* knowledge concerning the qualitative nature of the solution is available this may be used to guide effective computational schemes.

We shall be concerned with moving mesh numerical methods, which have the ability to adjust to the evolution of the solution (in order to track implicit moving boundaries and singularities for example), as well as to resolve sharp features and respect global properties. Such methods are therefore an attractive choice for problems of this type. The argument is reinforced in the case of scale-invariant problems for which both dependent and independent variables are strongly coupled. Fixed meshes are unable to replicate scale-invariance because they are time-independent. The coupling of independent and dependent variables is a recurrent theme in this paper and is used later on to motivate the development of a solution-adaptive moving mesh finite element method based on conservation.

Velocity-based moving mesh methods (also known as *Lagrangian methods* or in a wider context *Arbitrary Lagrangian Eulerian (ALE) methods*) rely on the construction of suitable velocities at points of the moving domain at each instant of time, as opposed to the construction of time-dependent mappings from a fixed computational domain to the moving domain [30,35,134]. The latter construction can be rather cumbersome in more than one dimension, and in any case the mappings need to be converted into velocities for incorporation into a time-dependent PDE. A velocity-based description, on the other hand, requires no formal reference to the computational domain and has the advantage that the velocity is available directly for incorporation into the time-dependent PDE. The evolution of the Lagrangian coordinate $\widehat{\mathbf{x}}(t)$ at time t follows from the velocity $\mathbf{v}(t,\mathbf{x})$ by integrating the ODE

$$\frac{d\widehat{\mathbf{x}}(t)}{dt} = \mathbf{v}(t,\widehat{\mathbf{x}}(t)), \tag{1.1}$$

where $\widehat{\mathbf{x}}(t)$ coincides instantaneously with the Eulerian coordinate $\mathbf{x}$ at the initial time.

The layout of the paper is as follows. In the next section (Section 2) we review time-dependent PDEs stated in both differential and integral form. Initially a fixed frame