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# Mixed Multiscale Finite Volume Methods for Elliptic Problems in Two-Phase Flow Simulations

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**Abstract.** We develop a framework for constructing mixed multiscale finite volume methods for elliptic equations with multiple scales arising from flows in porous media. Some of the methods developed using the framework are already known [20]; others are new. New insight is gained for the known methods and extra flexibility is provided by the new methods. We give as an example a mixed MsFV on uniform mesh in 2-D. This method uses novel multiscale velocity basis functions that are suited for using global information, which is often needed to improve the accuracy of the multiscale simulations in the case of continuum scales with strong non-local features. The method efficiently captures the small effects on a coarse grid. We analyze the new mixed MsFV and apply it to solve two-phase flow equations in heterogeneous porous media. Numerical examples demonstrate the accuracy and efficiency of the proposed method for modeling the flows in porous media with non-separable and separable scales.

**AMS subject classifications**: 65N30, 65N20, 34E13

**Key words**: Mixed multiscale finite volume methods, elliptic equations, two-phase flows, heterogeneous porous media.

## 1 Introduction

Subsurface flows are often affected by heterogeneities in a wide range of length scales. This causes significant challenges for subsurface flow modeling. Geological characterizations that capture these effects are typically developed at scales that are too fine for direct flow simulations. Usually, upscaled or multiscale models are employed for such systems.

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In upscaling methods, the original model is coarsened by numerically homogenizing parameters (e.g., permeability). The simulation is performed using the coarsened model, which may differ from the underlying fine-scale model. In multiscale methods, the fine-scale information is carried throughout the simulation and the coarse-scale equations are generally not expressed analytically, but rather formed and solved numerically.

Various numerical multiscale approaches for flows in porous media have been developed during the past decade. A multiscale finite element method (MsFEM) was introduced in [18] and takes its origin from the pioneering work [5]. Its main idea is to incorporate the small-scale information into finite element basis functions and capture their effect on the large scales via finite element computations. The MsFEM in [18] shares some similarities with a number of multiscale numerical methods, such as residual free bubbles [6], variational multiscale method [19], two-scale conservative subgrid approaches [3], heterogeneous multiscale method [15] and multiscale discontinuous Galerkin method [31]. Chen and Hou have applied the MsFEM idea in combination with a mixed finite element formulation to propose a mixed MsFEM [8]. Recently, Arbogast et al. [4] used domain decomposition approach and variational mixed formulation to develop a multiscale mortar mixed MsFEM. Jenny et al. [20] have used the ideas in [18] and finite volume framework to design a multiscale finite volume method (MsFV). The MsFV and its variants have proved successful in reservoir simulations.

Here we develop a framework for constructing mixed MsFV methods, which uses ideas from the mixed finite volume methods [24–26], multi-point flux approximations (MPFA) [2, 16], and mixed MsFEM. The mixed MsFV are mass conservative methods, which is an important property of the discretizations used in subsurface flow simulations (see [11] for related discussion). The important feature of the mixed finite volume methods is the direct approximation of the velocity, that is, specially constructed discrete spaces are used to approximate the velocity unknowns. We propose a novel way to construct multiscale velocity basis functions that are well suited for parallel computation. Mixed MsFEM reduces the system of coupled equations for pressure and velocity to a system only for the pressure. However, the reduction process is computational expensive and has some restrictions when the global mass matrix in mixed MsFEM is large. In the mixed MsFV, we compute the inverse of each local mass matrix instead of global mass matrix and get effective coarse-scale transmissibilities. This computation is cheap and efficient. In the MsFV proposed in [20], two sets of multiscale basis functions are computed: the first set of basis functions is to approximate pressure and the second set of basis functions is required to construct a conservative fine-scale velocity field. Only one set of multiscale basis functions is constructed in the mixed MsFV and the span of the basis functions are to approximate the velocity. Piecewise constant is used for pressure basis in the mixed MsFV. Hence the computation for basis functions in the mixed MsFV is less expensive than the MsFV. To the best of our knowledge, the mixed MsFV is a new numerical multiscale method.

Boundary effect is a great issue in many multiscale methods (e.g., [8, 18]). When we construct the multiscale basis functions in the mixed MsFV, we can use constant bound-