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High-Order Schemes Combining the Modified Equation Approach and Discontinuous Galerkin Approximations for the Wave Equation

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Abstract. We present a new high order method in space and time for solving the wave equation, based on a new interpretation of the "Modified Equation" technique. Indeed, contrary to most of the works, we consider the time discretization before the space discretization. After the time discretization, an additional biharmonic operator appears, which can not be discretized by classical finite elements. We propose a new Discontinuous Galerkin method for the discretization of this operator, and we provide numerical experiments proving that the new method is more accurate than the classical Modified Equation technique with a lower computational burden.

AMS subject classifications: 65M12, 65M60, 35L05

Key words: High order schemes, discontinuous Galerkin method, acoustic wave equation.

1 Introduction

Highly accurate solution of the full wave equation implies very high computational burdens. Indeed, to improve the accuracy of the numerical solution, one must considerably reduce the space step, which is the distance between two points of the mesh representing the computational domain. Obviously this will result in increasing the number of unknowns of the discrete problem. Besides, the time step, whose value fixes the number of required iterations for solving the evolution problem, is linked to the space step through the CFL (Courant-Friedrichs-Lewy) condition. The CFL number defines an upper bound for the time step in such a way that the smaller the space step is, the higher the number of iterations will be. In the three-dimensional case the problem can have more than ten million unknowns, which must be evaluated at each time-iteration. However,

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high-order numerical methods can be used for computing accurate solutions with larger space and time steps. Recently, Joly and Gilbert (cf. [7]) have optimized the Modified Equation Technique (MET), which was proposed by Shubin and Bell (cf. [11]) for solving the wave equation, and it seems to be very promising given some improvements. In this work, we apply this technique in a new way. Many works in the literature (see for instance [3, 5, 6, 11]) consider first the space discretization of the system before addressing the question of the time discretization. We intend here to invert the discretization process by applying first the time discretization using the MET and then to consider the space discretization. The time discretization causes high-order operators to appear (such as p-harmonic operators) and we have therefore to consider appropriate methods to discretize them. The Discontinuous Galerkin Methods are well adapted to this discretization, since they allow to consider piecewise discontinuous functions. In particular, using the Interior Penalty Discontinuous Galerkin (IPDG) method (see for instance [2, 4, 8] for the discretization of the Laplacian and [10] for the discretization of the biharmonic operator), one can enforce through the elements high-order transmission conditions, which are adapted to the high order operators to be discretized. The outline of this paper is as follows. In Section 2, we describe the classical application of the MET to the semi-discretized wave equation and we recall its properties. In Section 3, we obtain high-order schemes by applying this technique directly to the continuous wave equation and we present the numerical method we have chosen for the space discretization of the high order operators. In Section 4, we present numerical results to compare the performances of the new technique with the ones of the classical MET.

2 The modified equation technique

In this section, we recall the principle of the modified equation technique which allow us to obtain even order approximation in time and we refer to [6, 7, 11] for more details on this approach.

We consider here the acoustic wave equation in an heterogeneous bounded media $\Omega \subset \mathbb{R}^d$, $d=1,2,3$. For the sake of simplicity, we impose homogeneous Neumann boundary conditions on the Boundary $\Gamma := \partial\Omega$ but this study can be extended to Dirichlet boundary conditions without major difficulties.

$$\left\{\begin{array}{ll} \text{Find } u:\Omega\times[0,T]\mapsto\mathbb{R} \text{ such that :} & \\ \dfrac{1}{\mu(x)}\dfrac{\partial^2 u}{\partial t^2}-\operatorname{div}\left(\dfrac{1}{\rho(x)}\nabla u\right)=f, & \text{in } \Omega\times]0,T], \\ u(x,0)=u_0, \quad \dfrac{\partial u}{\partial t}(x,0)=u_1, & \text{in } \Omega, \\ \partial_n u=0, & \text{on } \partial\Omega, \end{array}\right. \tag{2.1}$$

where u stands for the displacement, μ is the compressibility modulus, ρ is the density and f is the source term. We assume that μ and ρ satisfy regularity conditions that we