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Pseudostress-Based Mixed Finite Element Methods for the Stokes Problem in $\mathbb{R}^n$ with Dirichlet Boundary Conditions. I: A Priori Error Analysis

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Abstract. We consider a non-standard mixed method for the Stokes problem in $\mathbb{R}^n$, $n \in \{2,3\}$, with Dirichlet boundary conditions, in which, after using the incompressibility condition to eliminate the pressure, the pseudostress tensor $\boldsymbol{\sigma}$ and the velocity vector $\mathbf{u}$ become the only unknowns. Then, we apply the Babuška-Brezzi theory to prove the well-posedness of the corresponding continuous and discrete formulations. In particular, we show that Raviart-Thomas elements of order $k \geq 0$ for $\boldsymbol{\sigma}$ and piecewise polynomials of degree k for $\mathbf{u}$ ensure unique solvability and stability of the associated Galerkin scheme. In addition, we introduce and analyze an augmented approach for our pseudostress-velocity formulation. The methodology employed is based on the introduction of the Galerkin least-squares type terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition for the velocity, all of them multiplied by suitable stabilization parameters. We show that these parameters can be chosen so that the resulting augmented variational formulation is defined by a strongly coercive bilinear form, whence the associated Galerkin scheme becomes well posed for any choice of finite element subspaces. For instance, Raviart-Thomas elements of order $k \geq 0$ for $\boldsymbol{\sigma}$ and continuous piecewise polynomials of degree $k+1$ for $\mathbf{u}$ become a feasible choice in this case. Finally, extensive numerical experiments illustrating the good performance of the methods and comparing them with other procedures available in the literature, are provided.

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1 Introduction

In the last decade there has been an increasing interest in new mixed finite element methods for linear and nonlinear Stokes problems. In particular, the velocity-pressure-stress formulation and its natural applicability to non-Newtonian flows has gained notoriety in recent years. Among the main strengths of this and other related mixed formulations, we highlight the fact that, besides the original unknowns, they provide direct approximations of several other variables of physical interest. In addition, the stress-based formulations yield a unified analysis for linear and nonlinear flows. However, the increase in the number of degrees of freedom of the resulting discrete systems and the symmetry requirement for the stress tensor constitute the main drawbacks of the approaches involving this unknown. In order to circumvent these disadvantages two important ideas have already been suggested in the literature. The first one, which goes back to [13] consists of imposing the symmetry of the stress in a weak sense through the introduction of a suitable Lagrange multiplier (rotation in elasticity and vorticity in fluid mechanics). The second one, which is more appealing nowadays, is given by the use of the pseudostress tensor instead of the stress in the corresponding setting of the Stokes equations.

As a consequence of the latter idea mentioned above, two new approaches for incompressible flows, namely the velocity-pressure-pseudostress and velocity-pseudostress formulations, arised specially in the context of least-squares and augmented methods (see, e.g. [5, 7, 12]). In fact, augmented mixed finite element methods for both pseudostress-based formulations of the stationary Stokes equations, which extend the results derived for the Lamé system in [15], are introduced and analyzed in [12]. The corresponding augmented mixed finite element schemes for the stress-based formulations of the Stokes problem, in which the vorticity is introduced as the Lagrange multiplier taking care of the weak symmetry of the stress, had been previously studied in [11]. Other related methods for the steady Stokes problem, based on least-squares formulations with two or three fields among velocity, velocity gradient, pressure, vorticity, stress, and pseudostress, can be found in [2, 3, 6, 9], and the references therein. Similarly, the extension of the results in [15] to the case of non-homogeneous Dirichlet boundary conditions in linear elasticity was provided in [14]. The use of the first Korn's inequality, as done in [15], is not possible in this case, and hence, an additional consistency term, determined precisely by the Dirichlet boundary condition, had to be incorporated into the augmented formulation. This extra term yielded the application of a modified Korn's inequality, which turned out to be crucial for the analysis in [14]. The results from [15] and [14] were extended in [17] to three-dimensional linear elasticity problems, while keeping the same advantages of the 2D case in the resulting augmented formulation.

Interestingly, the mixed finite element methods for the pure velocity-pseudostress formulation of the Stokes equations, that is without augmenting or employing least-squares terms, had not been studied in details until [8]. It is shown there that Raviart-Thomas elements of order $k \geq 0$ for the pseudostress and piecewise discontinuous polynomials of degree k for the velocity lead to a stable Galerkin scheme with quasi-optimal accuracy. The