Commun. Comput. Phys.
doi: 10.4208/cicp.260511.050811a

Vol. **12**, No. 1, pp. 1-41
July 2012

High Order Schemes on Three-Dimensional General Polyhedral Meshes — Application to the Level Set Method

Thibault Pringuey* and R. Stewart Cant

CFD Laboratory, Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK.

Received 26 May 2011; Accepted (in revised version) 5 August 2011

Available online 16 January 2012

Abstract. In this article, we detail the methodology developed to construct arbitrarily high order schemes — linear and WENO — on 3D mixed-element unstructured meshes made up of general convex polyhedral elements. The approach is tailored specifically for the solution of scalar level set equations for application to incompressible two-phase flow problems. The construction of WENO schemes on 3D unstructured meshes is notoriously difficult, as it involves a much higher level of complexity than 2D approaches. This due to the multiplicity of geometrical considerations introduced by the extra dimension, especially on mixed-element meshes. Therefore, we have specifically developed a number of algorithms to handle mixed-element meshes composed of convex polyhedra with convex polygonal faces. The contribution of this work concerns several areas of interest: the formulation of an improved methodology in 3D, the minimisation of computational runtime in the implementation through the maximum use of pre-processing operations, the generation of novel methods to handle complex 3D mixed-element meshes and finally the application of the method to the transport of a scalar level set.

AMS subject classifications: 65M08, 76-04, 76N99

Key words: WENO scheme, three-dimensional, unstructured mesh, mixed element, polyhedral element, hyperbolic equations, level set.

1 Introduction

The use of unstructured meshes in modern industrial CFD has dramatically increased over the years, following the development of more sophisticated numerical methods. The advantages of the unstructured approach are substantial: better resolution of the geometric details and significant reduction of meshing time when compared to structured-mesh

*Corresponding author. *Email addresses:* tp299@cam.ac.uk (T. Pringuey), rsc10@cam.ac.uk (R. S. Cant)

approaches on complex industrial geometries. The need for fine geometric definition is apparent in the simulation of industrially-relevant two-phase flow problems, for example in simulating primary break-up of the fuel spray in gas turbines. As shown in [13], accurate simulations of the atomisation process require precise numerical representation of the injection device. Modern modelling approaches to two-phase flows make use of level set concepts [4, 6, 9, 16, 21, 25, 27] and require the solution of a partial differential equation that represents the evolution of the level set scalar. As a result, the focus of this work has been placed on the development of a numerical method suitable for the transport of a level set on unstructured meshes.

Weighted Essentially Non-Oscillatory (WENO) schemes have become central to level set methods applied to the resolution of multiphase flows. Indeed, they offer the most efficient way to handle the large gradients incurred by the phase boundaries while maintaining a sharp interface. However, the construction of such schemes is much more complicated on unstructured meshes than on Cartesian grids as no particular direction can be identified in the distribution of the elements.

WENO schemes were originally developed for Cartesian grids [19, 24]. They were an evolution of the Essentially Non-Oscillatory (ENO) schemes introduced by Harten in 1987 [14, 15] to achieve high order accuracy and non-oscillatory properties near discontinuities such as shock waves in high-speed compressible flows. As the interest in unstructured meshes grew, WENO schemes were constructed for triangular [12, 17] and tetrahedral meshes [7, 8, 42].

These numerical schemes cope with the presence of discontinuities in the flow field by considering the solution on N_S stencils distributed around the targeted cell. Whereas the ENO methodology simply selects the stencil on which the solution is the smoothest, the WENO approach combines the data from the different stencils and weights their relative contributions according to the respective smoothness of the N_S datasets. As a result, WENO schemes reach higher orders of accuracy than ENO schemes at equal cost. In particular, Jiang showed that a WENO scheme constructed using r^{th} order ENO schemes could reach $(2r-1)^{\text{th}}$ order of accuracy in smooth regions of the flow [19].

The basic principles of the construction of WENO schemes for triangular meshes were presented by Friedrich [12], based on the work of Abgrall [1]. Later, Dumbser extended these ideas to tetrahedral meshes [7] and defined an approach to devise arbitrarily high order schemes. Titarev [36] improved this approach for two dimensional (2D) computational domains and produced high order schemes on mixed-element 2D meshes. We have extended the approach of Dumbser [7] and Titarev [36] to 3D mixed-element unstructured grids for linear hyperbolic equations [30]. Tsoutsanis [38] describes a similar method developed independently and has applied it to the solution of the Euler equations. In this paper, we present an extension of our scheme [30] to general polyhedral cells and apply it to the solution of the level set equation and the Burgers' equation.

It should be noted that many of the previous WENO schemes have been developed for application to high-speed flows, and specifically for the solution of systems of equations such as the Euler equations. This is not the aim of the present work, where at-