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A Numerical Method for Solving Elasticity Equations with Interfaces

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Abstract. Solving elasticity equations with interfaces is a challenging problem for most existing methods. Nonetheless, it has wide applications in engineering and science. An accurate and efficient method is desired. In this paper, an efficient non-traditional finite element method with non-body-fitting grids is proposed to solve elasticity equations with interfaces. The main idea is to choose the test function basis to be the standard finite element basis independent of the interface and to choose the solution basis to be piecewise linear satisfying the jump conditions across the interface. The resulting linear system of equations is shown to be positive definite under certain assumptions. Numerical experiments show that this method is second order accurate in the L^∞ norm for piecewise smooth solutions. More than 1.5th order accuracy is observed for solution with singularity (second derivative blows up) on the sharp-edged interface corner.

AMS subject classifications: 65N30

Key words: Elasticity equations, non-body fitted mesh, finite element method, jump condition.

1 Introduction

The importance of elasticity interface problems has been well recognized in a variety of disciplines. However, designing highly efficient methods for these problems is a difficult job, especially when the interface is not smooth.

Consider an open bounded domain $\Omega \subset R^d$. Let Γ be an interface of co-dimension $d-1$, which divides Ω into disjoint open subdomains, Ω^- and Ω^+, hence $\Omega = \Omega^- \bigcup \Omega^+ \bigcup \Gamma$. Assume that the boundary $\partial\Omega$ and the boundary of each subdomain $\partial\Omega^\pm$ are Lipschitz

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continuous as submanifolds. Since $\partial\Omega^{\pm}$ are Lipschitz continuous, so is Γ. A unit normal vector of Γ can be defined a.e. on Γ, see Section 1.5 in [10].

We seek solutions of the variable coefficient elliptic equation away from the interface Γ given by

$$\begin{cases} -\nabla\cdot(\beta_1(x)\nabla u_1(x))-\nabla\cdot(\beta_2(x)\nabla u_2(x))=f_1(x) \\ -\nabla\cdot(\beta_3(x)\nabla u_1(x))-\nabla\cdot(\beta_4(x)\nabla u_2(x))=f_2(x), \quad x\in\Omega\backslash\Gamma, \end{cases} \tag{1.1}$$

in which $x=(x_1,\cdots,x_d)$ denotes the spatial variables and ∇ is the gradient operator. The coefficient $\beta(x)$ is assumed to be a $d\times d$ matrix that is uniformly elliptic on each disjoint subdomain, Ω^- and Ω^+, and its components are continuously differentiable on each disjoint subdomain, but they may be discontinuous across the interface Γ. The right-hand side $f(x)$ is assumed to lie in $L^2(\Omega)$.

Given functions a and b along the interface Γ, we prescribe the jump conditions

$$\begin{cases} [u_1]_\Gamma(x)\equiv u_1^+(x)-u_1^-(x)=a_1(x), \\ [u_2]_\Gamma(x)\equiv u_2^+(x)-u_2^-(x)=a_2(x), \\ n\cdot(\beta_1^+(x)\nabla u_1^+(x)+\beta_2^+(x)\nabla u_2^+(x)) \\ \quad -n\cdot(\beta_1^-(x)\nabla u_1^-(x)+\beta_2^-(x)\nabla u_2^-(x))=b_1(x), \\ n\cdot(\beta_3^+(x)\nabla u_1^+(x)+\beta_4^+(x)\nabla u_2^+(x)) \\ \quad -n\cdot(\beta_3^-(x)\nabla u_1^-(x)+\beta_4^-(x)\nabla u_2^-(x))=b_2(x). \end{cases} \tag{1.2}$$

The "$\pm$" superscripts refer to limits taken from within the subdomains $\Omega^{\pm}$.

Finally, we prescribe boundary conditions

$$\begin{cases} u_1(x)=g_1(x), \quad x\in\partial\Omega, \\ u_2(x)=g_2(x), \quad x\in\partial\Omega, \end{cases} \tag{1.3}$$

for a given function g on the boundary $\partial\Omega$.

For simplicity, this paper discusses $d=2$ case. The three dimensional $d=3$ case is under investigation. The setup of the problem is illustrated in Fig. 1.

An elasticity system can be solved by both finite difference or finite element method. Due to the cross derivative term, usually the linear system of equations using the finite element formulation is better conditioned compared with that obtained using a finite difference discretization.

To solve the interface problem, first we need to generate a mesh. One approach is to use a body fitted mesh coupled with a finite element discretization, see for example, [1,3,5,6] for scalar elliptic partial differential equations. Recently, Cartesian meshes have become popular especially for moving interface problems to overcome the cost in the grid generation at every or every other time steps.