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# Radiogenic Source Identification for the Helium Production-Diffusion Equation

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**Abstract.** Knowledge of helium diffusion kinetics is critical for materials in which helium measurements are made, particulary for thermochronology. In most cases the helium ages were younger than expected, an observation attributes to diffusive loss of helium and the ejection of high energy alpha particles. Therefore it is important to accurately calculate the distribution of the source term within a sample. In this paper, the prediction of the helium concentrations as function of a spatially variable source term are considered. Both the forward and inverse solutions are presented. Under the assumption of radially symmetric geometry, an analytical solution is deduced based on the eigenfunction expansion. Two regularization methods, the Tikhonov regularization and the spectral cutoff regularization, are considered to obtain the regularized solution. Error estimates with optimal convergence order are shown between the exact solution and the regularized solution. Numerical examples are presented to illustrate the validity and effectiveness of the proposed methods.

**AMS subject classifications**: 65M32, 35Q80

**Key words**: Inverse source problem, production-diffusion equation, Tikhonov regularization.

## 1 Introduction

Helium isotopes are used extensively as thermochronometer in terrestrial and extraterrestrial materials [20]. He produced from radioactive decay of uranium and thorium

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series nuclides forms the basis of (U-Th)/He chronometry. The main attraction of helium for these applications is that its production rates are high compared to other isotope systems, coupled with the fact that high-precision, high-sensitivity helium analysis are comparatively easy. A critical consideration for these uses is that helium diffusion in most minerals occurs at moderate temperatures; failure to consider diffusive loss can lead to erroneously young helium-based age constraints. Knowledge of helium diffusion kinetics is therefore critical for materials in which helium measurements are made, particulary for thermochronology.

Despite of the appeal of (U-Th)/He system for thermochronological studies, one drawback of the technique is that spatial variations in radiogenic uranium and thorium in a sample can cause a non-uniform production of helium, and violate commonly made assumptions of a uniform source [9]. Spatial variations in uranium and thorium (often referred to a zoning of the parent isotopes) can produce substantial spatial fractionation of the parent/daughter ratio in accessory minerals likely to be used for helium thermochronometry. In the experience of apatite, zircon, titanite are commonly zoned and can limit the ultimate precision of helium ages [23]. Therefore it is important to accurately know the distribution of parent uranium and thorium isotopes (the source function) in a sample to be of use in helium dating. In this paper, the prediction of the a variable source term is formulated as an inverse radiogenic source problem.

Our model of helium production and diffusion considers the local helium concentration gradients resulting from ejection of high energy alpha particles from grain surfaces. It is assumed that the grain is of a spherical diffusion geometry, which is actually consistent with laboratory measurements of helium diffusion from apatite [22]. As a consequence of radiogenic production and diffusive loss, the concentration of helium as a function of the dimensional radial variable $\rho$ within the spherical diffusion domain of radius $R$ is [20]:

$$\frac{\partial u(t,\rho)}{\partial t}=a(t)\left[\frac{\partial^2 u(t,\rho)}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial u(t,\rho)}{\partial \rho}\right]+f(\rho),\quad 0<t<T,\quad 0<\rho<R, \tag{1.1}$$

where $a(t)$ is the time dependent diffusion coefficient, which is assumed to have a lower bound $a_0$ and an upper bound $a_1$, i.e.,

$$0<a_0\leq a(t)\leq a_1,$$

and $f(\rho)\geq 0$ corresponds to the spatial variable dependent radiogenic source production. The homogeneous initial condition is prescribed

$$u(0,\rho)=0,\quad 0<\rho<R. \tag{1.2}$$

The boundary conditions are given by

$$\lim_{\rho\to 0}u(t,\rho)\text{ bounded},\quad u(t,R)=0,\quad 0<t<T. \tag{1.3}$$