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Reduction of the Regularization Error of the Method of Regularized Stokeslets for a Rigid Object Immersed in a Three-Dimensional Stokes Flow

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Abstract. We focus on the problem of evaluating the velocity field outside a solid object moving in an incompressible Stokes flow using the boundary integral formulation. For points near the boundary, the integral is nearly singular, and accurate computation of the velocity is not routine. One way to overcome this problem is to regularize the integral kernel. The method of regularized Stokeslet (MRS) is a systematic way to regularize the kernel in this situation. For a specific blob function which is widely used, the error of the MRS is only of first order with respect to the blob parameter. We prove that this is the case for radial blob functions with decay property $\phi(r)=\mathcal{O}(r^{-3-\alpha})$ when $r\to\infty$ for some constant $\alpha>1$. We then find a class of blob functions for which the leading local error term can be removed to get second and third order errors with respect to blob parameter. Since the addition of these terms might give a flow field that is not divergence free, we introduce a modification of these terms to make the divergence of the corrected flow field close to zero while keeping the desired accuracy. Furthermore, these dominant terms are explicitly expressed in terms of blob function and so the computation time is negligible.

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Key words: Stokes flow, regularized Stokeslet, boundary integral equation, nearly singular integral.

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1 Introduction

Incompressible Newtonian Stokesian fluid flows, governed by the Stokes equations

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$$
\begin{aligned}
0 &= -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}, \\
0 &= \nabla \cdot \mathbf{u}
\end{aligned}
$$

have been widely used to study biological problems, for example, micro-organism locomotions [11], bioconvection [12], and effective viscosity of suspensions [7,8]. Here, $\mathbf{u}$ is the fluid velocity, p the pressure, $\mathbf{f}$ the external force or the body force, and μ the viscosity of the fluid. The equations are an approximation of the Navier-Stokes equations in the low Reynolds number regime.

When an object moves in and interacts with Stokes flow, the velocity of the exterior flow can be represented in terms of boundary integrals [13]. The boundary integral formulation has the advantage of reducing the full three-dimensional problem of solving for fluid flow to a two-dimensional problem of evaluating surface integrals. In the case of a rigid body immersed in a Stokes flow, we can represent the velocity as a first-kind integral equation with density equal to the traction on the surface and kernel the Stokeslet, the fundamental solution of the Stokes equations in free-space. The velocity can then be computed by numerically evaluating the surface integral. For evaluation points away from the surface, the integrand is smooth and slowly varying and we can use standard quadratures with high accuracy. But for points close to the surface, the integrand becomes nearly singular and accurate computation of the velocity is not routine. There are different approaches to this problem. One commonly used approach is to regularize the kernel.

The method of regularized Stokeslet (MRS), originally introduced by Cortez [5], is a systematic approach to regularize the kernel. The formulation is based on a free-space solution of the Stokes equation with concentrated but smooth forcing of the form

$$
\mathbf{f}(\mathbf{x}) = \mathbf{g}\phi^{\epsilon}(\mathbf{x}),
$$

where $\mathbf{f}$ is the force, $\mathbf{g}$ is a constant vector, $\phi^{\epsilon}(\mathbf{x})$ is a radially symmetric smooth approximation to the Dirac delta function, and ϵ is the blob parameter that controls the concentration of the force. In our work, we only consider blobs of the form

$$
\phi^{\epsilon}(\mathbf{x}) = \frac{1}{\epsilon^3}\phi\left(\frac{\mathbf{x}}{\epsilon}\right), \tag{1.1}
$$

where $\phi(\mathbf{x})$ is any radially symmetric function having integral 1 over $\mathbb{R}^3$. The velocity computed by the MRS is automatically divergence free. A boundary integral equation of the first kind based on a regularized Stokeslet together with the trapezoidal quadrature discretization was studied in [6]. In that work, the authors proved and showed by numerical examples that for the blob of the form (1.1) where

$$
\phi(\mathbf{x}) = \frac{15}{8\pi(|\mathbf{x}|^2+1)^{7/2}}, \tag{1.2}
$$