

## REVIEW ARTICLE

# Numerical Bifurcation Methods and their Application to Fluid Dynamics: Analysis beyond Simulation

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**Abstract.** We provide an overview of current techniques and typical applications of numerical bifurcation analysis in fluid dynamical problems. Many of these problems are characterized by high-dimensional dynamical systems which undergo transitions as parameters are changed. The computation of the critical conditions associated with

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these transitions, popularly referred to as ‘tipping points’, is important for understanding the transition mechanisms. We describe the two basic classes of methods of numerical bifurcation analysis, which differ in the explicit or implicit use of the Jacobian matrix of the dynamical system. The numerical challenges involved in both methods are mentioned and possible solutions to current bottlenecks are given. To demonstrate that numerical bifurcation techniques are not restricted to relatively low-dimensional dynamical systems, we provide several examples of the application of the modern techniques to a diverse set of fluid mechanical problems.

**AMS subject classifications:** 37H20, 35Q35, 76-02, 37M, 65P30

**Key words:** Numerical bifurcation analysis, transitions in fluid flows, high-dimensional dynamical systems.

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## 1 Introduction

Transition phenomena in flows of liquids and gases are of great fundamental interest. A prominent classical example is the sudden transition from laminar to turbulent flow which takes place in a circular pipe (Poiseuille flow) when the speed in the center of the pipe exceeds a critical value [51]. Other classical examples are the flow between rotating cylinders (Taylor-Couette flow), which undergoes successive transitions when the rotation rate of the inner cylinder is increased, and convection in a liquid layer heated from below (Rayleigh-Bénard-Marangoni flow), which shows a fascinating and rich set of flow patterns once a critical vertical temperature gradient is exceeded [62]. Central issues when studying these flows are the characterization of the range of conditions over which particular flows exist and the mechanisms of transition between the different flow patterns. In their classical treatise on Fluid Mechanics, [67] provide an elegant and general framework on the loss of stability of a general fluid flow due to changes in the parameters of the system.

Transitions in industrial and environmental flows are of great practical interest. Critical conditions in such flows, at which they may undergo a large qualitative change, are associated with what has been recently called a “tipping” point [48]. Examples are boundary flows, which may undergo qualitative changes in separation behavior and turbulence intensity, mean flows in turbulent buoyancy driven convection, which change