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Generalized Multiscale Finite Element Methods. Nonlinear Elliptic Equations

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Abstract. In this paper we use the Generalized Multiscale Finite Element Method (GMsFEM) framework, introduced in [26], in order to solve nonlinear elliptic equations with high-contrast coefficients. The proposed solution method involves linearizing the equation so that coarse-grid quantities of previous solution iterates can be regarded as auxiliary parameters within the problem formulation. With this convention, we systematically construct respective coarse solution spaces that lend themselves to either continuous Galerkin (CG) or discontinuous Galerkin (DG) global formulations. Here, we use Symmetric Interior Penalty Discontinuous Galerkin approach. Both methods yield a predictable error decline that depends on the respective coarse space dimension, and we illustrate the effectiveness of the CG and DG formulations by offering a variety of numerical examples.

AMS subject classifications: 35J60, 65N30

Key words: Generalized multiscale finite element method, nonlinear equations, high-contrast.

1 Introduction

Nonlinear partial differential equations represent a class of problems that have applications in many scientific communities [19,22,33,47,50]. Forchheimer flow, nonlinear elasticity, and electromagnetics are particular examples of physical processes that are modeled by nonlinear equations [3,19,33,35,50]. In addition to difficulties associated with the nonlinearity, these types of problems often involve coefficients that exhibit high-contrast,

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heterogeneous behavior. For example, when modeling subsurface flow, the underlying permeability field is often represented by a high-contrast coefficient in the pressure equation. One approach for solving a nonlinear equation is to linearize the problem and use an iterative method for obtaining the solution. For example, a Picard iteration yields an iterative process where a previous solution iterate is directly used in order to update the solution at the current iteration. In this case, a final solution is obtained when a suitable tolerance between the current and previous iteration is reached. While relatively easy to implement, iterative techniques typically require a repeated number of solves in order to obtain a convergent solution. In the case of a nonlinear elliptic equation, each iteration requires the numerical solution of a large system of equations that depends on the previous iterate. Thus, computing solutions on a fully resolved mesh quickly becomes a prohibitively expensive task. As such, techniques that allow for a more efficient computational procedure with a suitable level of accuracy are desirable.

The past few decades have seen the development of various multiscale solution techniques for capturing small scale effects on a coarse grid [1,7,30,38,39,41]. The multiscale finite element methods (MsFEM's) that we consider in this paper hinge on the construction of coarse spaces that are spanned by a set of independently computed multiscale basis functions. The multiscale basis functions are then coupled via a respective global formulation in order to compute the solution. In particular, solutions may be computed on a coarse grid while maintaining the fine-scale effects that are embedded into the basis functions. While standard multiscale methods have proven effective for a variety of applications (see, e.g., [29–31, 41]), in this paper we consider a more recent framework in which the coarse spaces may be systematically enriched to converge to the fine grid solution [9, 27, 28, 45]. More specifically, additional basis functions are chosen based on localized eigenvalue problems that capture the underlying behavior of the system. In this case, we may carefully choose the number of basis functions (and dimension of the coarse space) such that we achieve a desired level of accuracy. In this paper we additionally show that the systematic enrichment of coarse spaces is flexible with respect to the global formulation that is chosen to couple the resulting basis functions.

To treat the nonlinear elliptic equation considered in this paper we make use of the Generalized Multiscale Finite Element Method (GMsFEM) which was introduced in [26]. In order to do so, we apply a Picard iteration and treat an upscaled quantity of a previous solution iterate as a parameter in the problem. With this convention we follow an offline-online procedure in which the coarse space construction is split into two distinct stages; offline and online (see [12, 14, 20, 45, 49]). The main goal of this approach is to allow for the efficient construction of an online space (and an online solution) for each fixed parameter value and iteration. In the process, we precompute a larger-dimensional, *parameter-independent* offline space that accounts for an appropriate range of parameter values that may be used in the online stage. As construction of the offline space will constitute a one-time preprocessing step, only the online space will require additional work within the solution procedure. In the offline stage we first choose a fixed set of parameter values and generate an associated set of "snapshot" functions by solving localized prob-