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A Geometry-Preserving Finite Volume Method for Compressible Fluids on Schwarzschild Spacetime

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Abstract. We consider the relativistic Euler equations governing spherically symmetric, perfect fluid flows on the outer domain of communication of Schwarzschild spacetime, and we introduce a version of the finite volume method which is formulated from the geometric formulation (and thus takes the geometry into account at the discretization level) and is well-balanced, in the sense that it preserves steady solutions to the Euler equations on the curved geometry under consideration. In order to formulate our method, we first derive a closed formula describing all steady and spherically symmetric solutions to the Euler equations posed on Schwarzschild spacetime. Second, we describe a geometry-preserving, finite volume method which is based from the family of steady solutions to the Euler system. Our scheme is second-order accurate and, as required, preserves the family of steady solutions at the discrete level. Numerical experiments are presented which demonstrate the efficiency and robustness of the proposed method even for solutions containing shock waves and nonlinear interacting wave patterns. As an application, we investigate the late-time asymptotics of perturbed steady solutions and demonstrate its convergence for late time toward another steady solution, taking the overall effect of the perturbation into account.

AMS subject classifications: 76L05, 35L65

Key words: Compressible fluid, Euler system, Schwarzschild spacetime, geometry-preserving, finite volume method, steady state.

1 Introduction

The finite volume method is a versatile technique for scientific computing, which has found many applications in physical and engineering sciences. In particular, it allows

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one to approximate weak solutions (containing shock waves) to nonlinear hyperbolic systems of balance laws such as, for instance, the Euler equations of compressible fluid dynamics. In the present paper, we propose a geometry-preserving version of the finite volume method for general balance laws of hyperbolic partial differential equations, and we apply this method to the Euler equations for spherically symmetric, relativistic fluid flows posed on a curved spacetime and, for definiteness, the outer domain of communication of Schwarzschild spacetime. The proposed method is second-order accurate (in smooth regions of the flow), and well-balanced in the sense that steady solutions to the relativistic fluid equations are preserved at the discrete level.

The balance laws of interest in the present work have the following general form. Given a spacetime $(\mathcal{M},g)$ with Lorentzian metric g and covariant derivative operator ∇, we consider the class of balance laws

$$\nabla_\alpha\big(T^\alpha{}_\beta(\phi)\big)=0, \tag{1.1}$$

where $T^\alpha{}_\beta(\phi)$ represents the energy-momentum tensor of a set of (unknown) tensor fields ϕ defined on $\mathcal{M}$ (the indices α,β ranging between 0 and 3). We use a standard notation for the metric $g=g_{\alpha\beta}dx^\alpha dx^\beta$ in coordinates (x^α), and repeated indices are implicity summed up. We lower (or raise) indices with the metric $g_{\alpha\beta}$ (or its inverse $g^{\alpha\beta}$) so that, for instance, $u_\alpha=g_{\alpha\beta}u^\beta$ for a vector field u^α.

In particular, we are interested in the relativistic Euler equations for perfect compressible fluids, corresponding to $\phi=(\rho,u^\alpha)$ in (1.1) with

$$T^\alpha{}_\beta(\rho,u)=(\rho c^2+p)u^\alpha u_\beta+pg^\alpha{}_\beta. \tag{1.2}$$

Here, the scalar field $\rho\geq 0$ denotes the mass-energy density of the fluid and the vector field u^α its velocity, normalized so that $u^\alpha u_\alpha=-1$, while $c>0$ represents the light speed. Moreover, the pressure function in (1.2) is given by an equation of state $p=p(\rho)$ which must satisfy the (hyperbolicity) condition $p'(\rho)\in(0,c)$ (for all $\rho>0$), so that the equations (1.1) can be written in local coordinates as a system of nonlinear balance laws, which is strictly hyperbolic for $\rho>0$. In general, initially smooth solutions to (1.1)-(1.2) become discontinuous in finite time and shock waves form and then propagate within the spacetime.

A broad literature is available on the design of robust and accurate, shock-capturing schemes for general hyperbolic systems posed on a flat geometry like the Minkowski spacetime. In the present work, we intend to also take a *curved* background geometry into account, by following recent work by the first author and his collaborators; cf. [2–5, 15]. To this end, we introduce a finite volume scheme which is based on the geometric formulation (1.1), rather than on the corresponding partial differential equations in a specific local coordinate chart. In order to achieve the well-balanced property, we extend the approach in Russo et al. [23, 25, 26] and LeFloch et al. [14], and we introduce a discretization which accurately takes into account the family of steady solutions to the balance laws and, therefore, the geometric effects induced by the Lorentzian geometry