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A Kernel-Free Boundary Integral Method for Variable Coefficients Elliptic PDEs

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Abstract. This work proposes a generalized boundary integral method for variable coefficients elliptic partial differential equations (PDEs), including both boundary value and interface problems. The method is kernel-free in the sense that there is no need to know analytical expressions for kernels of the boundary and volume integrals in the solution of boundary integral equations. Evaluation of a boundary or volume integral is replaced with interpolation of a Cartesian grid based solution, which satisfies an equivalent discrete interface problem, while the interface problem is solved by a fast solver in the Cartesian grid. The computational work involved with the generalized boundary integral method is essentially linearly proportional to the number of grid nodes in the domain. This paper gives implementation details for a second-order version of the kernel-free boundary integral method in two space dimensions and presents numerical experiments to demonstrate the efficiency and accuracy of the method for both boundary value and interface problems. The interface problems demonstrated include those with piecewise constant and large-ratio coefficients and the heterogeneous interface problem, where the elliptic PDEs on two sides of the interface are of different types.

AMS subject classifications: 35J05, 65N06, 65N38

Key words: Elliptic partial differential equation, variable coefficients, kernel-free boundary integral method, finite difference method, geometric multigrid iteration.

1 Introduction

Variable coefficients elliptic partial differential equations (PDEs) appear in many important scientific and engineering applications such as the bidomain equations in compu-

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tational cardiology [19, 30, 45], the Cahn-Hilliard equation [7, 15, 52] and the Poisson-Boltzmann equation [32,50] in biophysics. Their accurate and efficient numerical method has long been an active research topic in computational physics [6,10,13,16,18,20–22,24, 25,35,36,44].

Among the numerical methods developed for variable coefficients elliptic partial differential equations (PDEs), those working with Cartesian grids have their advantages in grid generation as well as algorithm efficiency and solution accuracy, and are often regarded to be more suitable for free boundary and moving interface problems than others, which are based on unstructured grids.

In this paper, we propose a new Cartesian grid based boundary integral method for both boundary value and interface problems of variable coefficients elliptic partial differential equation (PDE). The solution to the boundary value problem is computed by iteratively solving corresponding interface problems. The elliptic interface problem of our interest consists of the PDEs (2.5)-(2.6), the interface conditions (2.7) and the boundary condition (2.8) on the boundary of a rectangle. The PDEs (2.5)-(2.6) in the interface problem may be heterogeneous in the sense that the Green functions associated with the elliptic operators on two sides of the interface are in general different. For this, we call the interface problem (2.5)-(2.8) as a heterogeneous interface problem.

When the coefficients of the PDE are piecewise constant, the interface problem (2.5)-(2.8) can be solved by a fast Fourier transform (FFT) based iterative method [26]. Here we remark that the approach in [26] can only solve the special piecewise constant interface problem where the reaction coefficients simultaneously vanish or the ratio of them equals that of the diffusion coefficients. When the boundary condition (2.8) is further replaced by a far field boundary condition in the free space, the piecewise constant coefficients interface problem can be formulated as a boundary integral equation (BIE) and solved by a boundary integral method [47]. The standard boundary integral method requires an analytic expression of the kernel and the corresponding Green function. In general, for variable coefficients elliptic PDE, it is difficult if not impossible to find an analytical expression for the Green function. Even for constant coefficients elliptic PDE, if it is defined on a bounded domain and subject to a non-periodic boundary condition, the Green function is usually difficult to find, too. For this reason, the standard boundary integral method is traditionally restricted to the constant coefficients elliptic boundary value problem or the piecewise constant coefficients interface problem in a rectangle domain subject to the periodic boundary condition or in the free space subject to the far field radiation condition.

Other related works using Cartesian grid based methods include ones for boundary value problems [3, 4, 8, 11, 12, 17, 23, 31, 33, 34, 40] and those for interface problems [9, 25, 27–29, 37–39, 41, 46, 51]. All these methods are based on standard finite difference or finite element discretization. Near the boundary or interface, the discretization has to be modified in order to maintain desired accuracy. Most of the methods modify the coefficient matrix of the discrete linear system, which often makes it difficult to directly apply the geometric multigrid or FFT based fast solver. They have to resort to algebraic