Commun. Comput. Phys.
doi: 10.4208/cicp.120313.230813s

Vol. **15**, No. 4, pp. 959-980
April 2014

Finite Volume Hermite WENO Schemes for Solving the Hamilton-Jacobi Equation

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Received 12 March 2013; Accepted (in revised version) 23 August 2013

Available online 21 January 2014

Abstract. In this paper, we present a new type of Hermite weighted essentially non-oscillatory (HWENO) schemes for solving the Hamilton-Jacobi equations on the finite volume framework. The cell averages of the function and its first one (in one dimension) or two (in two dimensions) derivative values are together evolved via time approaching and used in the reconstructions. And the major advantages of the new HWENO schemes are their compactness in the spacial field, purely on the finite volume framework and only one set of small stencils is used for different type of the polynomial reconstructions. Extensive numerical tests are performed to illustrate the capability of the methodologies.

AMS subject classifications: 65M06, 65M99, 35L65

Key words: HWENO scheme, finite volume, Hamilton-Jacobi equation.

1 Introduction

In this paper, we investigate using the finite volume Hermite weighted essentially non-oscillatory (HWENO) reconstruction methodologies for directly solving the Hamilton-Jacobi (H-J) equations:

$$\begin{cases} \phi_t + H(\nabla\phi) = 0, & (x_1,\cdots,x_n,t) \in \Omega\times[0,\infty), \\ \phi(x_1,\cdots,x_n,0) = \phi_0(x_1,\cdots,x_n), & (x_1,\cdots,x_n)\in\Omega, \end{cases} \tag{1.1}$$

where $\nabla\phi = (\phi_{x_1},\cdots,\phi_{x_n})^T$.

The Hamilton-Jacobi equations are often used in geometric optics, computer vision, material science, image processing and variational calculus [4, 13, 21]. Yet, the solutions to (1.1) are continuous but their derivatives are discontinuous. And such solutions may

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not be unique unless using the physical implications and then getting the viscosity solutions [1]. It is well known that the HJ equations are closely related to conservation laws, hence we can obtain the exact solutions of HJ equations from those of conservation laws, respectively, and successful numerical methods for conservation laws can be adapted for solving the HJ equations. Along this line we mention the early works, Osher and Sethian [14] proposed a second order essentially non-oscillatory (ENO) scheme and Osher and Shu [15] presented high order ENO schemes to solve the Hamilton-Jacobi equations. Then, a high order of Weighted ENO (WENO) scheme was proposed by Jiang and Peng [7]. Recently, Qiu [16, 17] and with Shu [20] also proposed Hermite WENO schemes for solving the Hamilton-Jacobi equations on structured meshes. In 1996, Lafon and Osher [10] constructed the ENO schemes for solving the Hamilton-Jacobi equations on unstructured meshes. Zhang and Shu [25], Li and Chan [12] further developed high order WENO schemes for solving two dimensional Hamilton-Jacobi equations by using the nodal based weighted essentially non-oscillatory algebraic polynomial reconstructions on triangular meshes. And some finite element methods for arbitrary triangular meshes were developed in [2, 3, 6, 11]. Unlike the discontinuous Galerkin (DG) method of Hu and Shu [6] which applied DG framework on the conservation law system satisfied by the derivatives of the solution, Cheng and Shu presented DG methods to directly solve HJ equations (1.1) for ϕ in [5] and new flux was presented to keep stability of the method. In [24], a new DG method to directly solve HJ equations (1.1) was presented, in which local DG method was applied to approximate derivatives of ϕ.

This is a continuation paper for solving the Hamilton-Jacobi equations [7,16,17,20,25], following the (H)WENO methodologies for the conservation laws [18, 19, 26, 27]. We evolve both the cell averages of the viscosity solution ϕ and its derivatives over the target cell. Both the cell average of the solution and the cell averages of its derivatives are used to reconstruct the point values of the solution ϕ and its derivatives at different Gauss-Lobatto quadrature points on the target cell and its boundaries, respectively. Comparing with the original WENO schemes of Jiang and Peng [7], Qiu [16], Li and Chan [12], one major advantage of HWENO schemes [17, 20] is its compactness in the reconstructions, since both the solution and its derivatives are evolved in time. Also, the new HWENO schemes are more compact than the original HWENO schemes, e.g., [17, 20], easily in using the same one set of spacial stencils in the reconstructions and purely on the finite volume framework.

The organization of this paper is as follows: in Section 2, we review and construct the new finite volume HWENO schemes in 1D and 2D in detail for solving Hamilton-Jacobi equations and present extensive numerical results in Section 3 to verify the accuracy and stability of these approaches. Concluding remarks are given in Section 4.

2 The construction of HWENO schemes for the Hamilton-Jacobi equations