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Dimension-Reduced Hyperbolic Moment Method for the Boltzmann Equation with BGK-Type Collision

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Abstract. We develop the dimension-reduced hyperbolic moment method for the Boltzmann equation, to improve solution efficiency using a numerical regularized moment method for problems with low-dimensional macroscopic variables and high-dimensional microscopic variables. In the present work, we deduce the globally hyperbolic moment equations for the dimension-reduced Boltzmann equation based on the Hermite expansion and a globally hyperbolic regularization. The numbers of Maxwell boundary condition required for well-posedness are studied. The numerical scheme is then developed and an improved projection algorithm between two different Hermite expansion spaces is developed. By solving several benchmark problems, we validate the method developed and demonstrate the significant efficiency improvement by dimension-reduction.

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Key words: Dimension-reduction, moment system, NRxx, global hyperbolicity, boundary condition, microflow.

1 Introduction

In kinetic theory, gas flows are often categorized into different regimes by the Knudsen number Kn. The classic Navier-Stokes-Fourier (NSF) equations are adequate to model the behavior of fluid in the continuum regime ($Kn < 0.001$), and can capture certain flow features in the slip regime ($0.001 < Kn < 0.1$) if the various velocity-slip technique and

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temperature-jump boundary conditions are used. However, in rarefied fluids or microflows, the gas flows are in the transition regime ($Kn > 0.1$) and the NSF equations fail [20].

It is generally accepted that the Boltzmann equation can capture the physics of gases in transition regimes. Several methods have been devised to solve the Boltzmann equation directly, e.g., direct simulation Monte Carlo (DSMC) due to Bird [2], discrete velocity methods (DVM) [10, 12, 13, 25, 33], and the unified scheme for BGK and Shakhov models [17,31,32]. On the other hand, Grad [14] developed a class of models called the method of moments, to approximate the Boltzmann equation. Based on Grad's idea, a number of moment equations were put forward [5, 15, 16, 18, 26, 29]. In [5], the authors proposed a general numerical method, referred as NRxx method, for numerically solving the moment equations to arbitrary order. In the transition regime, it is well-known that the traditional no-slip wall boundary conditions do not apply, and the Boltzmann equation with the Maxwell boundary conditions [24] can depict the gas state-cf. [20,21] for details. In [6], the Maxwell boundary condition for the NRxx method is studied, and [8,9] further demonstrate the effectiveness of NRxx method in rarefied fluids simulation.

However, one flaw of Grad type moment equations is that they are not hyperbolic, which makes the system not well-posed and limits the application of the NRxx method. Recently, a new regularization was developed to make Grad's moment system globally hyperbolic [3, 4], and results in a class of hyperbolic moment equations (HME). A corresponding numerical scheme was proposed in [7], and the Maxwell boundary condition for the NRxx method in [6] can be extended to the HME without any modification. Given this progress both theoretically and numerically, the HME method is a valid and effective method to solve the Boltzmann equation for rarefied fluids and microflows for the one-dimensional case [7].

For a two-dimensional problem, the dimension of the molecular velocity is still 3. In [10], the author has proposed a dimension-reduction technique, by introducing an auxiliary distribution function to reduce the dimension of the molecular velocity the same as that of the space. As is well-known, this technique can reduce the computational cost in numerical simulation.

In the present work, we consider the two-dimensional case, and use the dimension-reduction technique on the Boltzmann equation to get two coupled dimension-reduced Boltzmann equations, and then invoke the NRxx method to achieve a dimension-reduced NRxx method. The globally hyperbolic regularization in [3] is first derived for the dimension-reduced NRxx method to obtain a class of dimension-reduced hyperbolic moment equations (DRHME). Both the BGK model and Shakhov model are studied for the collision term. The Maxwell boundary condition is also studied. In the numerical scheme, classic time splitting is adopted to handle the convection term and the collision term in the dimension-reduced Boltzmann equations, and the finite volume method is applied on the convection part.

During the derivation of the DRHME, the dimension-reduction technique is applied on the Boltzmann equation to get two reduced distribution functions and two coupled