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High-Order Symplectic Schemes for Stochastic Hamiltonian Systems

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Abstract. The construction of symplectic numerical schemes for stochastic Hamiltonian systems is studied. An approach based on generating functions method is proposed to generate the stochastic symplectic integration of any desired order. In general the proposed symplectic schemes are fully implicit, and they become computationally expensive for mean square orders greater than two. However, for stochastic Hamiltonian systems preserving Hamiltonian functions, the high-order symplectic methods have simpler forms than the explicit Taylor expansion schemes. A theoretical analysis of the convergence and numerical simulations are reported for several symplectic integrators. The numerical case studies confirm that the symplectic methods are efficient computational tools for long-term simulations.

AMS subject classifications: 60H10, 65C30, 65P10

Key words: Stochastic Hamiltonian systems, symplectic integration, mean-square convergence, high-order schemes.

1 Introduction

The symplectic integration covers a special type of numerical methods which are capable of preserving the symplecticity properties of the Hamiltonian system. The pioneering work on the symplectic integration is due to de Vogelaere [1], Ruth [2] and Kang Feng [3]. Symplectic methods have been applied successfully to deterministic Hamiltonian systems, and numerical simulations consistently show that the most important feature of this approach is that the accuracy of the computed solution is guaranteed even for long

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term computation [4,5]. In this paper, we study the symplectic numerical integration for stochastic Hamiltonian systems, and propose an approach to generate symplectic numerical schemes of any desired order.

Consider the stochastic differential equations (SDEs) in the sense of the Stratonovich:

$$dP_i = -\frac{\partial H^{(0)}(P,Q)}{\partial Q_i}dt - \sum_{r=1}^{m}\frac{\partial H^{(r)}(P,Q)}{\partial Q_i}\circ dw_t^r, \qquad P(t_0)=p, \tag{1.1a}$$

$$dQ_i = \frac{\partial H^{(0)}(P,Q)}{\partial P_i}dt + \sum_{r=1}^{m}\frac{\partial H^{(r)}(P,Q)}{\partial P_i}\circ dw_t^r, \qquad Q(t_0)=q, \tag{1.1b}$$

where P, Q, p, q are n-dimensional vectors with components P^i, Q^i, p^i, q^i, $i=1,\cdots,n$ and w_t^r, $r=1,\cdots,m$ are independent standard Wiener Processes. The SDEs (1.1) are called the Stochastic Hamiltonian System (SHS) (see [6]). The SHS (1.1) includes both Hamiltonian systems with additive or multiplicative noise.

A non-autonomous SHS is given by time-dependent Hamiltonian functions $H^{(r)}(t,P,Q)$, $r=0,\cdots,m$. However, it can be rewritten as an autonomous SHS by introducing new variables e_k and f_k. Let

$$df_r = dt, \quad de_r = -\frac{\partial H^{(r)}(t,P,Q)}{\partial t}\circ dw_t^r, \quad (\text{where } dw_t^0 := dt),$$

with the initial condition $e_r(t_0) = -H^{(r)}(t_0,p,q)$ and $f_r(t_0) = t_0$, $r=0,\cdots,m$. Then the new Hamiltonian functions $\bar{H}^{(r)}(\bar{P},\bar{Q}) = H^{(r)}(f_r,P,Q)$, $r=1,\cdots,m$, and $\bar{H}^{(0)}(\bar{P},\bar{Q}) = H^{(0)}(f_r,P,Q)+e_0+\cdots+e_m$, define an autonomous SHS with $\bar{P}=(P^T,e_0,\cdots,e_m)^T$ and $\bar{Q}=(Q^T,f_0,\cdots,f_m)^T$. Hence, in this study, we will only investigate the autonomous case as given in (1.1).

There are growing interests and efforts on the theoretical study and computational implementation of numerical methods for SHS [6,8–11]. Milstein et al. [6,8] introduced the symplectic numerical schemes to SHS, and demonstrated the superiority of the symplectic methods for long time computation. Although they proposed symplectic schemes of orders two or three for special types of SHS, for the general SHS with multiplicative noise given in (1.1), they construct only symplectic schemes of mean square order 0.5. In this paper we apply an approach based on generating functions and we construct symplectic schemes of arbitrary high mean square order. Hong et al. [9] developed a predictor-corrector scheme for a linear SDE with an additive noise, a simple case of SHS. In [11], Wang et al. proposed the variational integrators to construct the stochastic symplectic schemes.

The generating functions associated with the SHS (1.1) were rigorously introduced in [15]. Recently, Wang et al. [12–14] proposed generating functions methods to construct symplectic schemes for SHS. But, in those papers, only the product of one-fold Stratonovich integrals is considered, so their approach cannot be used to construct