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Multi-Symplectic Fourier Pseudospectral Method for the Kawahara Equation

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Abstract. In this paper, we derive a multi-symplectic Fourier pseudospectral scheme for the Kawahara equation with special attention to the relationship between the spectral differentiation matrix and discrete Fourier transform. The relationship is crucial for implementing the scheme efficiently. By using the relationship, we can apply the Fast Fourier transform to solve the Kawahara equation. The effectiveness of the proposed methods will be demonstrated by a number of numerical examples. The numerical results also confirm that the global energy and momentum are well preserved.

AMS subject classifications: 65M06, 65M70, 65T50, 65Z05, 70H15

Key words: Kawahara equation, Multi-symplecticity, Fourier pseudospectral method, FFT.

1 Introduction

In this paper, we consider the Kawahara equation [1]

$$2\frac{\partial u}{\partial t}+\alpha\frac{\partial^3 u}{\partial x^3}+\beta\frac{\partial^5 u}{\partial x^5}=\frac{\partial}{\partial x}f(u,u_x,u_{xx}), \tag{1.1}$$

where $u(x,t)$ is a scalar function, $\alpha,\beta\neq 0$ are real parameters and $f(u,u_x,u_{xx})$ is a smooth function. Eq. (1.1) is a model equation for plasma waves, capillary-gravity waves and other dispersive phenomena when the cubic KdV-type dispersion is weak. The form of (1.1) which occurs most often in applications is with $f(u,u_x,u_{xx})=au^2$, where a is a nonzero constant. Eq. (1.1) was first proposed by Kawahara [2] in 1972, as a model

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equation describing solitary-wave propagation in dispersive media. By studying systematically Eq. (1.1) with $f(u,u_x,u_{xx})=-3u^2$, Kawahara observed that the solitary wave states could have oscillatory tails, and computed examples of such waves numerically. A more general nonlinearity was derived for water waves by Olver [3], using Hamiltonian perturbation theory, with further generalization given by Craig and Groves [4]. Existence and uniqueness of solutions to nonlinear Kawahara equations are obtained in [5].

As far as we know, numerical methods for this equation are very limited. Yuan, Shen and Wu [6] developed a Dual-Petrov-Galerkin method for the equation and showed some excellent numerical results. In Ref. [7], Hu and Deng developed a multi-symplectic Preissmann scheme. In this paper, we aim to develop a new multi-symplectic method for the Kawahara equation.

Many PDEs could be written as multi-symplectic Hamiltonian PDEs [8]

$$M\mathbf{z}_t+K\mathbf{z}_x=\nabla_{\mathbf{z}}S(\mathbf{z}), \tag{1.2}$$

where $\mathbf{z}(x,t)\in\mathbb{R}^n(n\geq 3)$, M and K are skew-symmetric matrices, and $S(\mathbf{z})$ is a smooth function. It is well known that Eq. (1.2) has multi-symplectic conservation law

$$\frac{\partial}{\partial t}\omega+\frac{\partial}{\partial x}\kappa=0, \tag{1.3}$$

where $\omega=\frac{1}{2}d\mathbf{z}\wedge Md\mathbf{z}$, $\kappa=\frac{1}{2}d\mathbf{z}\wedge Kd\mathbf{z}$. As the multi-symplectic conservation law is a signification geometric property of the Hamiltonian PDEs, numerical integrators which can preserve corresponding discrete multi-symplectic conservation law are expected. Bridges and Reich [9, 10] called such integrators are multi-symplectic integrators. Many equations were constructed as multi-symplectic Hamiltonian PDEs and integrated by some multi-symplectic methods (please refer to review paper [11]). These methods include multi-symplectic Preissmann scheme [9], multi-symplectic Fourier pseudospectral method [12, 13], multi-symplectic wavelet collocation method [14, 15], multi-symplectic Euler box scheme [16–19], multi-symplectic splitting method [20, 21] and so on. A great many numerical experiments show that multi-symplectic methods perform better than traditional numerical methods in long time simulations.

Bridges and Reich [12] suggested the idea of multi-symplectic spectral discretization on Fourier space. Based on their theory, Chen and Qin [13] proposed multi-symplectic Fourier pseudospectral (MSFP) method for Hamiltonian PDEs and applied it to integrate nonlinear Schrödinger (NLS) equation with periodic boundary conditions. Then, Wang [23] made some numerical analysis for the NLS equation. Later, the MSFP method was widely applied to other equations [14, 22, 24, 25] and so on. The key of the MSFP method is the spectral differentiation matrix (SDM) which can be obtained easily by proposed method in Ref. [13]. However, it needs a lot of storage space and a large amount of calculations to apply SDM directly, especially when the number of the nodes is large. In this paper, we develop a relationship between the SDM and discrete Fourier transform (DFT). By the relationship, we can apply Fast Fourier transform (FFT) easily in numerical