Commun. Comput. Phys.
doi: 10.4208/cicp.280813.190214a

Vol. **16**, No. 2, pp. 440-466
August 2014

On Time-Splitting Pseudospectral Discretization for Nonlinear Klein-Gordon Equation in Nonrelativistic Limit Regime

Xuanchun Dong[1], Zhiguo Xu[3,1] and Xiaofei Zhao[2,*]

[1] *Beijing Computational Science Research Center, Beijing 100084, P.R. China.*
[2] *Department of Mathematics, National University of Singapore, Singapore 119076, Singapore.*
[3] *College of Mathematics, Jilin University, Changchun 130012, P.R. China.*

Received 28 August 2013; Accepted (in revised version) 19 February 2014

Available online 23 May 2014

Abstract. In this work, we are concerned with a time-splitting Fourier pseudospectral (TSFP) discretization for the Klein-Gordon (KG) equation, involving a dimensionless parameter $\varepsilon\in(0,1]$. In the nonrelativistic limit regime, the small ε produces high oscillations in exact solutions with wavelength of $\mathcal{O}(\varepsilon^2)$ in time. The key idea behind the TSFP is to apply a time-splitting integrator to an equivalent first-order system in time, with both the nonlinear and linear subproblems exactly integrable in time and, respectively, Fourier frequency spaces. The method is fully explicit and time reversible. Moreover, we establish rigorously the optimal error bounds of a second-order TSFP for fixed $\varepsilon=\mathcal{O}(1)$, thanks to an observation that the scheme coincides with a type of trigonometric integrator. As the second task, numerical studies are carried out, with special efforts made to applying the TSFP in the nonrelativistic limit regime, which are geared towards understanding its temporal resolution capacity and meshing strategy for $\mathcal{O}(\varepsilon^2)$-oscillatory solutions when $0<\varepsilon\ll 1$. It suggests that the method has uniform spectral accuracy in space, and an asymptotic $\mathcal{O}(\varepsilon^{-2}\Delta t^2)$ temporal discretization error bound (Δt refers to time step). On the other hand, the temporal error bounds for most trigonometric integrators, such as the well-established Gautschi-type integrator in [6], are $\mathcal{O}(\varepsilon^{-4}\Delta t^2)$. Thus, our method offers much better approximations than the Gautschi-type integrator in the highly oscillatory regime. These results, either rigorous or numerical, are valid for a splitting scheme applied to the classical relativistic NLS reformulation as well.

AMS subject classifications: 35L70, 65M12, 65M15, 65M70

Key words: Klein-Gordon equation, high oscillation, time-splitting, trigonometric integrator, error estimate, meshing strategy.

*Corresponding author. *Email addresses:* dong.xuanchun@gmail.com (X. Dong), zhxfnus@gmail.com (X. Zhao), xuzg11@csrc.ac.cn (Z. Xu)

1 Introduction

The relativistic Klein-Gordon (KG) equation in d-dimensions ($d=1,2,3$) reads, under a proper non-dimensionalization [6, 26–28, 30, 31, 43],

$$\varepsilon^2\partial_{tt}u(\mathbf{x},t)-\Delta u(\mathbf{x},t)+\frac{1}{\varepsilon^2}u(\mathbf{x},t)+f(u(\mathbf{x},t))=0, \qquad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \tag{1.1a}$$

with initial conditions:

$$u(\mathbf{x},0)=\phi_1(\mathbf{x}), \qquad \partial_t u(\mathbf{x},0)=\frac{1}{\varepsilon^2}\phi_2(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d. \tag{1.1b}$$

The KG equation is also known as the relativistic version of the Schrödinger equation and used to describe the motion of a spinless particle; see, e.g. [13, 34] for its derivation. In this work, $u=u(\mathbf{x},t)$ is considered to be a real-valued scalar field, the dimensionless parameter $\varepsilon>0$ is inversely proportional to the speed of light, ϕ_1 and ϕ_2 are two given real-valued functions independent of ε. $f(\cdot)$ is a real-valued function describing the nonlinear interaction, independent of ε and satisfying $f(0)=0$. The KG equation (1.1) is time symmetry or time reversible and conserves the *energy*, provided that $u(\cdot,t)\in H^1(\mathbb{R}^d)$ and $\partial_t u(\cdot,t)\in L^2(\mathbb{R}^d)$,

$$\begin{aligned} E(t)&:=\int_{\mathbb{R}^d}\left[\varepsilon^2(\partial_t u(\mathbf{x},t))^2+|\nabla u(\mathbf{x},t)|^2+\frac{1}{\varepsilon^2}(u(\mathbf{x},t))^2+F(u(\mathbf{x},t))\right]\mathrm{d}\mathbf{x}\\ &\equiv\int_{\mathbb{R}^d}\left[\frac{1}{\varepsilon^2}(\phi_2(\mathbf{x}))^2+|\nabla\phi_1(\mathbf{x})|^2+\frac{1}{\varepsilon^2}(\phi_1(\mathbf{x}))^2+F(\phi_1(\mathbf{x}))\right]\mathrm{d}\mathbf{x}:=E(0), \quad t\geq 0, \end{aligned} \tag{1.2}$$

with $F(u)=2\int_0^u f(\rho)\,\mathrm{d}\rho$, $u\in\mathbb{R}$.

When $\varepsilon>0$ in (1.1) is fixed, e.g. $\varepsilon=1$, corresponding to the $\mathcal{O}(1)$-speed of light regime, a surge of analysis and numerics results have been reported in literatures. For instance, the Cauchy problem was considered in [2, 10, 23, 24, 38]. In particular, global existence of solutions was established in [10] for $F(u)\geq 0$ (defocusing case); and possible blow-up was shown in [2] for $F(u)<0$ (focusing case). For more results in this regime, we refer the readers to [29, 33, 36, 40] and references given therein. Along the numerical aspect, many numerical schemes have been proposed in literatures. The classical numerical methods are the standard finite difference time domain methods including energy conservative, semi-implicit and explicit finite difference discretizations [1, 15, 25, 32, 41] and some other approaches such as finite element or spectral discretization in space coupled with appropriate time integrator, like standard finite difference or Gautschi-type exponential integrator [6, 12, 14, 42]. For comparisons of different numerical methods, we refer the readers to [6, 22, 32].

Over the past decade, more attentions have been paid to the regime $0<\varepsilon\ll 1$ in (1.1), which corresponds to the nonrelativistic limit or the speed of light goes to infinity. In this regime, the analysis and efficient simulation are mathematically rather complicated