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A Flexible Boundary Procedure for Hyperbolic Problems: Multiple Penalty Terms Applied in a Domain

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Abstract. A new weak boundary procedure for hyperbolic problems is presented. We consider high order finite difference operators of summation-by-parts form with weak boundary conditions and generalize that technique. The new boundary procedure is applied near boundaries in an extended domain where data is known. We show how to raise the order of accuracy of the scheme, how to modify the spectrum of the resulting operator and how to construct non-reflecting properties at the boundaries. The new boundary procedure is cheap, easy to implement and suitable for all numerical methods, not only finite difference methods, that employ weak boundary conditions. Numerical results that corroborate the analysis are presented.

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Key words: Summation-by-parts, weak boundary conditions, penalty technique, high-order accuracy, finite difference schemes, stability, steady-state, non-reflecting boundary conditions.

1 Introduction

High order finite difference methods provide an efficient approach for problems in computational science. The efficiency can be used either to increase the accuracy for a fixed

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number of mesh points or to reduce the computational cost for a given accuracy by reducing the number of mesh points [27, 48]. The main drawback with high order finite difference methods is the complicated boundary treatment required to obtain a stable method.

Finite difference operators which satisfy the summation-by-parts (SBP) property [28, 29, 42], are central difference operators in the interior domain augmented with special stencils near the domain boundaries. These SBP operators in combination with weak well-posed boundary conditions lead to energy stability [6, 8, 16, 19, 31, 40, 41]. One such boundary treatment is the simultaneous approximation term (SAT) method [7], which linearly combines the partial differential equation to be solved with well-posed boundary conditions [5, 8, 34, 39].

In this paper we will extend this technique by applying the boundary conditions in an extended domain. As an introduction, consider the continuous one-dimensional right going ($a>0$) advection problem

$$u_t+au_x=0, \quad 0\le x\le 1, \quad t>0, \tag{1.1}$$

with a boundary condition $u(0,t)=g_0(t)$ at $x=0$ for well-posedness. The energy method applied to (1.1) yields the following continuous energy rate

$$\frac{d}{dt}\|u\|^2=au(0,t)^2-au(1,t)^2, \tag{1.2}$$

where $\|u\|^2=\int_0^1 u^2dx$. By letting $u(0,t)=g_0(t)$, well-posedness follows.

Let the approximative solution at grid point x_i be denoted u_i, and the discrete solution vector $\mathbf{u}^T=[u_0,u_1,\cdots,u_N]$. A finite difference approximation of (1.1) using an SBP operator with SAT treatment for the boundary condition can be written as

$$\mathbf{u}_t+aP^{-1}Q\mathbf{u}=P^{-1}\alpha_{00}(u_0-g_0)e_0, \tag{1.3}$$

where the difference operator $P^{-1}Q$ approximates d/dx, P is symmetric and positive definite, $Q+Q^T=B=diag(-1,0,\cdots,0,1)$, α_{00} is called the penalty coefficient and $e_0=[1,0,\cdots,0]^T$ is the unit vector that positions the penalty term at $i=0$. The discrete energy method on (1.3) gives

$$\frac{d}{dt}\|\mathbf{u}\|_P^2=(a+2\alpha_{00})u_0^2-2\alpha_{00}u_0g_0-au_N^2, \tag{1.4}$$

where $\|\mathbf{u}\|_P^2=\mathbf{u}^TP\mathbf{u}$. Clearly, for $\alpha_{00}\le-(a/2)$, we have a bounded energy. Without imposing boundary conditions ($\alpha_{00}=0$), the rate (1.4) mimics (1.2) perfectly. (A boundary condition at $x=0$ is necessary for stability, and it will be imposed below.) For more details using this technique, see [2, 7, 8, 31, 42].

The SAT technique (weak imposition of boundary condition or penalty technique) is normally applied only at one grid point (as in the example above) [2, 3, 22, 23, 31, 36, 37,