

# Simulation of Propagating Acoustic Wavefronts with Random Sound Speed

Sheri L. Martinelli\*

*Undersea Warfare Weapons, Vehicles, and Defensive Systems Department,  
Naval Undersea Warfare Center, Newport, Rhode Island 02841, USA.*

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**Abstract.** A method for simulating acoustic wavefronts propagating under random sound speed conditions is presented. The approach applies a level set method to solve the Eikonal equation of high frequency acoustics for surfaces of constant phase, instead of tracing rays. The Lagrangian nature often makes full-field ray solutions difficult to reconstruct. The level set method captures multiple-valued solutions on a fixed grid. It is straightforward to represent other sources of uncertainty in the input data using this model, which has an advantage over Monte Carlo approaches in that it yields an expression for the solution as a function of random variables.

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**Key words:** Level set method, wavefront methods, high-frequency acoustic propagation, stochastic collocation.

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## 1 Introduction

Modeling and simulation of underwater sound propagation in shallow water environments is critical to the design of acoustic systems and system performance evaluation. Although not a full substitute for in-water operational testing, access to high fidelity simulation can drastically reduce the need for expensive, risky, in-water experiments. One such family of applications involves high frequency active arrays operating in environments where reverberation dominates the noise field. Transmit frequencies at least on the order of 10 kHz render full wave equation simulations computationally impractical due to the need for high mesh resolution in space and time.

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\*Corresponding author. *Email address:* sheri\_martinelli@alumni.brown.edu (S. L. Martinelli)

For this reason, it is common to apply the geometric acoustics (geometric optics) approximation, c.f. Chapter 3 in [1], to the linear wave equation for the acoustic pressure,

$$p_{tt} - c(\mathbf{x})^2 \Delta p = 0, \quad (1.1)$$

$$p(t, \mathbf{x}) \approx e^{i\omega S(t, \mathbf{x})} \sum_{k=0}^{\infty} A_k(t, \mathbf{x}) (i\omega)^{-k}, \quad (1.2)$$

to yield an Eikonal equation for the acoustic phase:

$$S(t, \mathbf{x}) \pm c|\nabla S(t, \mathbf{x})| = 0, \quad (1.3)$$

and a transport equation for the first amplitude term:

$$(A_0)_t + c(\mathbf{x}) \frac{\nabla S \cdot \nabla A_0}{|\nabla S|} + \frac{c(\mathbf{x})^2 \Delta S - S_{tt}}{2c(\mathbf{x})|\nabla S|} A_0 = 0. \quad (1.4)$$

Eq. (1.2) is based on a WKB expansion about the large parameter  $\omega$ . The standard computational approach to solving the nonlinear equation (1.3) is ray tracing, which is based on the method of characteristics. This is a Lagrangian method: the user specifies initial conditions for a number of rays, i.e., location and starting angles from an initial wavefront, and solves for the resulting trajectories defined by the system with Hamiltonian  $H(\mathbf{x}, \mathbf{k}) = c(\mathbf{x})|\mathbf{k}|$ , where  $\mathbf{k}$  is the generalized momentum vector. Examples of applications that rely on ray tracing for system design and performance prediction include acoustic tomography [2] and underwater communications [3]. In acoustic tomography, accurate travel time data between a source and receiver over long ranges are used to infer information about ocean currents and temperatures. Both applications have a need for sorting out multi-path time arrivals that occur as a result of ray bending, and surface and bottom scattering. The ray approach has many known limitations. In particular, the Lagrangian nature of the model leads to difficulty resolving wave arrivals at a fixed location in space since the user does not have control over the spatial discretization beyond the source. That is, small perturbations in the ray shooting angle can lead to significantly differing trajectories. This is especially true in range-dependent, shallow water environments where reflections from rough surfaces may scatter the trajectories further. Hence there is benefit to consider fixed frame of reference approaches. Toward that end, [4] proposes applying the level set method [5, 6] to the problem of high frequency underwater acoustic propagation. The level sets approach solves the Eikonal equation (1.3) in the phase space  $(\mathbf{x}, \mathbf{k})$  by representing the initial wavefront as the zero level set of a vector-valued function  $\Phi(t, \mathbf{x}, \mathbf{k})$  in the higher-dimensional space. The wavefronts propagate according to the velocity field determined by the local ray direction, but on a fixed spatial grid. One recovers the wavefront at time  $t$  by projecting the zero level set of  $\Phi(t, \mathbf{x}, \mathbf{k})$  back into the physical space.

The intent of this work is to extend the method outlined in [4] to generate realizations of propagating wavefronts in the presence of random perturbations in the sound