Dirichlet-to-Neumann Mapping for the Characteristic Elliptic Equations with Symmetric Periodic Coefficients

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Abstract. Based on the numerical evidences, an analytical expression of the Dirichlet-to-Neumann mapping in the form of infinite product was first conjectured for the one-dimensional characteristic Schrödinger equation with a sinusoidal potential in [Commun. Comput. Phys., 3(3): 641-658, 2008]. It was later extended for the general second-order characteristic elliptic equations with symmetric periodic coefficients in [J. Comp. Phys., 227: 6877-6894, 2008]. In this paper, we present a proof for this Dirichlet-to-Neumann mapping.

AMS subject classifications: 65M99, 81-08
Key words: Dirichlet-to-Neumann mapping, Schrödinger equation, symmetric periodic potentials, absorbing boundary conditions.

1 Introduction

Periodic structure problems largely exist in the science and engineering such as semiconductor nanostructures, semiconductor superlattices [3, 24], photonic crystals (PC) structures [2, 15, 18], meta materials [20] and Bragg gratings of surface plasmon polariton (SPP) waveguides [11, 19]. Usually they are modeled by partial differential equations with periodic coefficients and/or periodic geometries. In order to numerically solve these equations efficiently, one usually confines the computational domain by introducing artificial boundaries and imposing suitable boundary conditions on them. For wave-like equations, the ideal boundary conditions should not only lead to well-posed problems, but also mimic the perfect absorption of waves which travel out of the computational domain...
through the artificial boundaries. Right in this sense, these boundary conditions are usually called absorbing (or transparent, non-reflecting in the same spirit) in the literature.

The study of absorbing boundary conditions (ABCs) for linear wave-like equations has been a hot research topic for many years and significant developments have been made on their designing and implementing. The interested reader is referred to the review papers [1, 8, 10, 23]. Comparatively, the study of exact or approximate ABCs for periodic structure problems is relatively a very current research topic, cf. the recent papers [6, 7, 14, 21, 22, 25, 26]. For a review on the theory and numerical techniques of waves in locally periodic media we refer the reader to [4, 9, 12].

Based on the numerical evidences, one of the authors [28] conjectured an exact ABC in the form of Dirichlet-to-Neumann (DtN) mapping for the characteristic Schrödinger equation

\[ \left[-\partial^2_x + V(x)\right] y = z y, \quad (1.1) \]

when the potential \( V \) is sinusoidal. This DtN mapping is expressed as an infinite product

\[ \frac{y'(0)}{y(0)} = -\sqrt{-z + \mu_{0,V}^{NN}} \prod_{m=1}^{\infty} \frac{1}{\sqrt{-z + \mu_{m,V}^{NN}}} \quad \Im z > 0, \quad (1.2) \]

where \( \{\mu_{r,V}^{NN}\}_{r \geq 0} \) are eigenvalues of the Schrödinger operator \( -\partial^2_x + V(x) \) with Neumann boundary conditions imposed on the periodic cell boundaries, and \( \{\mu_{r,V}^{DD}\}_{r \geq 1} \) are eigenvalues with Dirichlet boundary conditions imposed. This DtN mapping was then extended in [5] for the general second-order characteristic elliptic equations

\[ \left[-\partial_x m^{-1}(x) \partial_x + V(x)\right] y = \rho(x) z y, \quad \forall x \geq 0, \quad (1.3) \]

where \( \rho, V \) and \( m \) are supposed to be symmetric periodic. The exact DtN mapping for (1.3) was conjectured to be

\[ \frac{y'(0)}{y(0)} = -\sqrt{c(0) \rho(0)} \prod_{m=1}^{\infty} \frac{1}{\sqrt{-z + \mu_{m,V}^{NN}}} \quad \Im z > 0, \quad (1.4) \]

where again, \( \{\mu_{v,V}^{NN}\}_{v \geq 0} \) are the eigenvalues of characteristic elliptic equations (1.3) with Neumann boundary conditions and \( \{\mu_{v,V}^{DD}\}_{v \geq 1} \) with Dirichlet boundary conditions specified on the periodic cell boundaries. The exact DtN mapping (1.4) was used in [5] to design the exact ABCs for a set of one-dimensional time-dependent wave equations with symmetric periodic coefficients.

In this paper, we give a proof for the DtN mapping expression (1.4). Our idea was stimulated by the work of J. Pöschel and E. Trubowitz. In [17], they considered the characteristic Schrödinger equation (1.1) with 1-periodic potentials. For the basic solution \( \varphi \)