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A Multi-Domain Hybrid DG and WENO Method for Hyperbolic Conservation Laws on Hybrid Meshes

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Abstract. In [SIAM J. Sci. Comput., 35(2)(2013), A1049–A1072], a class of multi-domain hybrid DG and WENO methods for conservation laws was introduced. Recent applications of this method showed that numerical instability may encounter if the DG flux with Lagrangian interpolation is applied as the interface flux during the moment of conservative coupling. In this continuation paper, we present a more robust approach in the construction of DG flux at the coupling interface by using WENO procedures of reconstruction. Based on this approach, such numerical instability is overcome very well. In addition, the procedure of coupling a DG method with a WENO-FD scheme on hybrid meshes is disclosed in detail. Typical testing cases are employed to demonstrate the accuracy of this approach and the stability under the flexibility of using either WENO-FD flux or DG flux at the moment of requiring conservative coupling.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Discontinuous Galerkin method, weighted essentially nonoscillatory scheme, hybrid method, conservation laws.

1 Introduction

To maintain reliability of solution, high order methods with low diffusion and low dissipation have become necessary when the structure of solution is complex and the resolution of traditional second order methods is not satisfactory. With this regard, high order methods have been widely developed in solving nonlinear hyperbolic conservation laws in recent years.

The first type is the high order finite difference (FD) schemes, for instance, the weighted essentially non-oscillatory finite difference (WENO-FD) schemes [7, 8]. The

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obvious advantages of a WENO-FD schemes are highly efficient and easy to achieve high order accuracy in structure meshes. It has usually been used in the direct numerical simulation (DNS) of turbulence or in the area of computational aeroacoustics (CAA) where high order accuracy plays a significant role. Nevertheless, the difficulty in handling complex geometries for this type of schemes limits their practical application. The second type of high order methods belongs to high order finite volume (FV) schemes, such as the weighted essentially non-oscillatory finite volume (WENO-FV) schemes [9] and high order k-exact finite volume methods [10]. This type of methods has flexibility in handling almost arbitrary mesh and geometry which makes them dominant in the CFD community. However, when it extends to high order accuracy, the implementation of a FV type schemes usually becomes much involved since the number of cells used for reconstruction increases tremendously when the accuracy order goes higher. The last type is the high order discontinuous Galerkin (DG) type methods, which include the traditional Runge-Kutta discontinuous Galerkin (RKDG) methods developed by Cockburn and Shu [1–3], the Spectral volume/difference (SV/SD) methods introduced by Wang et al. [19–21] and the correction procedure via reconstruction (CPR) schemes recently introduced by Huynh [22]. The DG type methods can fit for complex geometries in a more flexible way and are compact as each element only communicates with its immediate face-neighbors through approximate Riemann solvers. In DG type methods, one important component is the construction of nonlinear limiters, see related work [4–6] for reference. Unfortunately, this type of methods also suffers some not well solved issues and weaknesses, such as high computational cost and difficulties in developing more reliable nonlinear limiters. These become major bottlenecks for DG methods in practical applications.

All of those high order methods mentioned above have their advantages and disadvantages. In order to overcome the disadvantages of a specific type of methods, hybrid methods which combine the advantages of two different kinds of high order methods have been proposed. There are mainly two kinds of hybrid approaches presented in literatures for solving Euler equations, one is based on local polynomial reconstruction, the other is based on computational domain decomposition.

Balsara et al. [11] adopted the first approach and presented a novel class of hybrid RKDG and Hermite WENO schemes on structured grids. Their hybrid algorithm stores cell averages as well as slopes for each cell of the RKDG methods and reconstructs high order degrees of freedom through Hermite reconstruction. Luo et al. [12, 13] introduced a reconstructed DG method (RDG) for Euler equations on arbitrary grids. In contrast to the traditional RKDG methods, RDG reconstructs high order degrees of freedom using a least-squares technique. The idea behind this hybrid approach is to combine the efficiency of the reconstruction methods widely used in FV methods and the accuracy and robustness of DG methods. Dumbser et al. [14] presented a unified framework for constructing one step finite volume and discontinuous Galerkin schemes on unstructured meshes, resulting in a class of $P_N P_M$ schemes. This approach yields two special cases: classical high order finite volume methods ($N=0$) and DG methods ($N=M$). Very