A Nominally Second-Order Cell-Centered Finite Volume Scheme for Simulating Three-Dimensional Anisotropic Diffusion Equations on Unstructured Grids

Pascal Jacq[1], Pierre-Henri Maire[1,*] and Rémi Abgrall[2]

[1] *CEA/CESTA, 15 Avenue des Sablières CS 60001, 33116 Le Barp cedex, France.*
[2] *Institüt für Mathematik, Universität Zürich, CH-8057 Zürich, Switzerland.*

Abstract. We present a finite volume based cell-centered method for solving diffusion equations on three-dimensional unstructured grids with general tensor conduction. Our main motivation concerns the numerical simulation of the coupling between fluid flows and heat transfers. The corresponding numerical scheme is characterized by cell-centered unknowns and a local stencil. Namely, the scheme results in a global sparse diffusion matrix, which couples only the cell-centered unknowns. The space discretization relies on the partition of polyhedral cells into sub-cells and on the partition of cell faces into sub-faces. It is characterized by the introduction of sub-face normal fluxes and sub-face temperatures, which are auxiliary unknowns. A sub-cell-based variational formulation of the constitutive Fourier law allows to construct an explicit approximation of the sub-face normal heat fluxes in terms of the cell-centered temperature and the adjacent sub-face temperatures. The elimination of the sub-face temperatures with respect to the cell-centered temperatures is achieved locally at each node by solving a small and sparse linear system. This system is obtained by enforcing the continuity condition of the normal heat flux across each sub-cell interface impinging at the node under consideration. The parallel implementation of the numerical algorithm and its efficiency are described and analyzed. The accuracy and the robustness of the proposed finite volume method are assessed by means of various numerical test cases.

*Corresponding author. *Email addresses:* `pascal.jacq@inria.fr` (P. Jacq), `maire@celia.u-bordeaux1.fr` (P.-H. Maire), `remi.abgrall@math.uzh.ch` (R. Abgrall)

1 Introduction

In this paper, we describe a finite volume scheme to solve anisotropic diffusion equations on unstructured grids. This three-dimensional scheme is the natural extension of the two-dimensional scheme CCLAD (Cell-Centered LAgrangian Diffusion) initially presented in [28]. Let us mention that it is also a 3D extension of the scheme proposed by Le Potier [34]. We aim at developing a robust and flexible method for diffusion operators devoted to the numerical modeling of the coupling between heat transfers and fluid flows. More precisely, we are concerned by the numerical simulation of heat transfers in the domain of hypersonic ablation of thermal protection systems [12]. In this context, one has to solve not only the compressible Navier-Stokes equations for the fluid flow but also the anisotropic heat equation for the solid materials which compose the thermal protection. These two models, *i.e.*, the Navier-Stokes equations and the heat equation, are strongly coupled by means of a surface ablation model which describes the removal of surface materials resulting from complex thermochemical reactions such as sublimation. We point out that in our case, the Navier-Stokes equations are solved employing a cell-centered finite volume method and the thermal protection system consists of several distinct materials with discontinuous conductivity tensors. This leads to the following requirements related to the diffusion scheme under consideration:

- It should be a finite volume scheme wherein the primary unknown, *i.e.*, the temperature is located at the cell center.

- It should be a sufficiently accurate and robust scheme to cope with unstructured three-dimensional grids composed of tetrahedral and/or hexahedral cells.

Before describing the main features of our finite volume scheme, let us briefly give an overview of the existing cell-centered diffusion scheme on three-dimensional grids. The simpler cell-centered finite volume is the so-called two-point flux approximation wherein the normal component of the heat flux at a cell interface is computed using the finite difference of the adjacent temperatures. It is well known that this method is consistent if and only if the computational grid is orthogonal with respect to the metrics induced by the symmetric positive definite conductivity tensor. This flaw renders this method inoperative for solving anisotropic diffusion problems on three-dimensional unstructured grids. It has motivated the work of Aavatsmark and his co-authors to develop a class of finite volume schemes based on multi-point flux approximations (MPFA) for solving the elliptic flow equation encountered in the context of reservoir simulation, refer to [2,3]. In this method, the flux is approximated by a multi-point expression based on transmissibility coefficients. These coefficients are computed using the pointwise continuity of the normal flux and the temperature across cell interfaces. The link between lowest-order mixed finite element and multi-point finite volume methods on simplicial meshes is investigated in [40]. The class of MPFA methods is characterized by cell-centered unknowns and a local stencil. The global diffusion matrix corresponding to this type of schemes