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An Efficient Two-Grid Scheme for the Cahn-Hilliard Equation

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Abstract. A two-grid method for solving the Cahn-Hilliard equation is proposed in this paper. This two-grid method consists of two steps. First, solve the Cahn-Hilliard equation with an implicit mixed finite element method on a coarse grid. Second, solve two Poisson equations using multigrid methods on a fine grid. This two-grid method can also be combined with local mesh refinement to further improve the efficiency. Numerical results including two and three dimensional cases with linear or quadratic elements show that this two-grid method can speed up the existing mixed finite method while keeping the same convergence rate.

AMS subject classifications: 65N30, 65N55, 68W25

Key words: Cahn-Hilliard equation, two-grid, adaptivity, mixed method, multigrid.

1 Introduction

We consider numerical solutions of the following Cahn-Hilliard (C-H) equation

$$\begin{cases} \dfrac{\partial u}{\partial t}-\Delta(-\varepsilon\Delta u+F'(u))=0, & x\in\Omega, \\ u(x,0)=u_0(x), & x\in\Omega, \\ \partial_n u=\partial_n(-\varepsilon\Delta u+F'(u))=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^d (d=2,3)$ is a bounded domain, n denotes the unit outward normal of the boundary $\partial\Omega$, $\varepsilon > 0$ is a small but positive constant, and $F(u)$ is a given energy potential. The solution $u(x,t)$ can represent the difference between two concentration, and in most applications $u \in [-1,1]$.

The C-H equation describes the process of phase separation, first introduced by Cahn and Hilliard in the late 1950s [4–6]. Numerical methods for solving the C-H equations provide an important tool on the studying of the dynamics of the C-H equation.

One main difficulty of numerical methods for the C-H equation is the discretization of the fourth order differential operator. For the rectangular domain, finite difference methods or spectral methods [24, 25] can be used. For unstructured grids of a general domain with possible complex geometry structure, finite element methods seems a better choice. However, it is well known that conforming finite-element spaces for fourth order equations is not easy to construct especially in three dimensions. Possible remedy includes non-conforming elements [14, 49] or discontinuous Galerkin methods [1, 39, 43]. Here we consider mixed finite element methods (MFEs) [15, 17, 18], which can give the numerical approximation not only to the concentration u but also to the chemical potential $w = \phi(u) - \epsilon\Delta u$. The price of using the mixed formulation is that the nonlinear system is in the saddle point form which is in general bigger and harder to solve than the symmetric positive definite system obtained by a conforming or non-conforming discretization. In our two grid method to be presented later, we shall overcome this shortcoming.

Another focus of developing accurate and efficient numerical scheme is the energy stable time discretization. It is shown that the implicit Euler method applied to the C-H equation is unconditionally stable and obey the energy law [1]. For full implicit schemes, however, a nonlinear system should be solved at every time step which is usually ten times slower than the first order semi-implicit schemes for which fast Poisson solvers can be applied, see [35]. The semi-implicit scheme (F(u) being explicit and $\Delta^2 u$ being implicit) is conditionally energy stable but with a restrictive constraint for the time step. To remove the requirement of small time step, stabilization [8, 35] or convex splitting scheme [16,27,40] can be applied. Other energy stable schemes can be found in [40,43,49]. Especially the stability and the convergence of mixed finite element methods for the C-H equation was investigated in [17, 18]. We shall not explore more on the stability in this paper.

Instead we are interested in efficient ways to improve the accuracy of numerical approximations. In this paper, we shall apply the two-grid method [44,45] to the C-H equation. The main idea of two-grid methods is solving the C-H equation using a stable mixed finite element method on a coarse grid first, then solving two Poisson equations on the fine grid. The nonlinear system on the coarse grid, because of the small size of the system, can be solved without too much computational cost. On the fine gird, we only need to solve two decoupled Poisson equations, which can be solved efficiently by off-the-shelf solvers such as the multigrid solvers. We shall demonstrate that the two-grid method can achieve the same convergence rate as the standard implicit mixed finite element method on the fine grid but with less computational time.