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A Finite-Volume Method for Nonlinear Nonlocal Equations with a Gradient Flow Structure

José A. Carrillo[1], Alina Chertock[2] and Yanghong Huang[1,*]

[1] *Department of Mathematics, Imperial College London, London SW7 2AZ, UK.*
[2] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA.*

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Abstract. We propose a positivity preserving entropy decreasing finite volume scheme for nonlinear nonlocal equations with a gradient flow structure. These properties allow for accurate computations of stationary states and long-time asymptotics demonstrated by suitably chosen test cases in which these features of the scheme are essential. The proposed scheme is able to cope with non-smooth stationary states, different time scales including metastability, as well as concentrations and self-similar behavior induced by singular nonlocal kernels. We use the scheme to explore properties of these equations beyond their present theoretical knowledge.

AMS subject classifications: 65M08, 35R09, 34B10

Key words: Finite-volume method, gradient flow, nonlinear nonlocal equations.

1 Introduction

In this paper, we consider a finite-volume method for the following problem:

$$\begin{cases} \rho_t = \nabla \cdot \left[\rho \nabla \left(H'(\rho) + V(\mathbf{x}) + W * \rho \right) \right], & \mathbf{x} \in \mathbb{R}^d,\, t > 0, \\ \rho(\mathbf{x},0) = \rho_0(\mathbf{x}), \end{cases} \tag{1.1}$$

where $\rho(\mathbf{x},t) \geq 0$ is the unknown probability measure, $W(\mathbf{x})$ is an interaction potential, which is assumed to be symmetric, $H(\rho)$ is a density of internal energy, and $V(\mathbf{x})$ is a confinement potential.

Equations such as (1.1) appear in various contexts. If W and V vanishes, and $H(\rho) = \rho\log\rho - \rho$ or $H(\rho) = \rho^m$, it is the classical heat equation or porous medium/fast diffusion

*Corresponding author. *Email addresses:* carrillo@imperial.ac.uk (J. A. Carrillo), chertock@math.ncsu.edu (A. Cherock), yanghong.huang@imperial.ac.uk (Y. Huang)

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equation [1]. If mass-conserving, self-similar solutions of these equations are sought, the quadratic term $V(\mathbf{x}) = |\mathbf{x}|^2$ is added, leading to new equations in similarity variables. More generally, V usually appears as a confining potential in Fokker-Planck type equations [2,3]. Finally, W is related to the interaction energy, and can be as singular as the Newtonian potential in chemotaxis system [4] or as smooth as $W(\mathbf{x}) = |\mathbf{x}|^\alpha$ with $\alpha > 2$ in granular flow [5].

The free energy associated to Eq. (1.1) is given by (see [6–8]):

$$E(\rho)=\int_{\mathbb{R}^d} H(\rho)d\mathbf{x}+\int_{\mathbb{R}^d} V(\mathbf{x})\rho(\mathbf{x})d\mathbf{x}+\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y})d\mathbf{x}d\mathbf{y}. \tag{1.2}$$

This energy functional is the sum of internal energy, potential energy and interaction energy, corresponding to the three terms on the right-hand side of (1.2), respectively. A simple computation shows that, at least for classical solutions, the time-derivative of $E(\rho)$ along solutions of (1.1) is

$$\frac{d}{dt}E(\rho)=-\int_{\mathbb{R}^d}\rho|\mathbf{u}|^2 d\mathbf{x}:=-I(\rho), \tag{1.3}$$

where

$$\mathbf{u}=-\nabla\xi, \quad \xi:=\frac{\delta E}{\delta\rho}=H'(\rho)+V(\mathbf{x})+W*\rho. \tag{1.4}$$

The functional I will henceforth be referred to as the entropy dissipation functional.

Eq. (1.1) and its associated energy $E(\rho)$ are the subjects of intensive study during the past fifteen years, see e.g. [6,8–10] and the references therein. The general properties of (1.1) are investigated in the context of interacting gases [6,8,9], and are common to a wide variety of models, including granular flows [5,11–13], porous medium flows [2,3], and collective behavior in biology [14]. The gradient flow structure, in the sense of (1.3), is generalized from smooth solutions to measure-valued solutions [10]. Certain entropy-entropy dissipation inequalities between $E(\rho)$ and $I(\rho)$ are also recognized to characterize the fine details of the convergence to steady states [2,3,6].

The steady state of (1.1), if it exists, usually verifies the form

$$\xi=H'(\rho)+V(\mathbf{x})+W*\rho=C, \quad \text{on supp } \rho, \tag{1.5}$$

where the constant C could be different on different connected components of supp ρ. In many cases, especially in the presence of the interaction potential W, there are multiple steady states, whose explicit forms are available only for particular W. Most of studies of these steady states are based on certain assumptions on the support and the characterizing equation (1.5).

In this work, we propose a positivity preserving finite-volume method to treat the general nonlocal nonlinear PDE (1.1). Moreover, we show the existence of a discrete free