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Relaxation Schemes for the M_1 Model with Space-Dependent Flux: Application to Radiotherapy Dose Calculation

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Abstract. Because of stability constraints, most numerical schemes applied to hyperbolic systems of equations turn out to be costly when the flux term is multiplied by some very large scalar. This problem emerges with the M_1 system of equations in the field of radiotherapy when considering heterogeneous media with very disparate densities. Additionally, the flux term of the M_1 system is non-linear, and in order for the model to be well-posed the numerical solution needs to fulfill conditions called realizability. In this paper, we propose a numerical method that overcomes the stability constraint and preserves the realizability property. For this purpose, we relax the M_1 system to obtain a linear flux term. Then we extend the stencil of the difference quotient to obtain stability. The scheme is applied to a radiotherapy dose calculation example.

AMS subject classifications: 82C40, 35L65, 65M25

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1 Introduction

The present work is devoted to the numerical solution of a moment system of equations, which describes the transport of electrons in tissues. The model finds application in the

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field of radiotherapy dose calculation when considering low density media [11]:

$$\frac{1}{\rho(x)}\nabla_x.\psi^1(x,\epsilon)=\partial_\epsilon(S(\epsilon)\psi^0)(x,\epsilon), \tag{1.1a}$$

$$\frac{1}{\rho(x)}\nabla_x.\psi^2(x,\epsilon)=\partial_\epsilon(S(\epsilon)\psi^1)(x,\epsilon)-2T(\epsilon)\psi^1(x,\epsilon), \tag{1.1b}$$

where the unknowns $\psi^0\in\mathbb{R}$, $\psi^1\in\mathbb{R}^3$ and $\psi^2\in\mathbb{R}^{3\times3}$ depend on energy $\epsilon\in\mathbb{R}^+$ and position $x\in\mathbb{R}^3$. The stopping power $S>0$ and the transport coefficient $T\geq0$ are functions of ϵ characterizing the loss of energy and the deflection of the electrons during their transport. Finally, $\rho(x)>0$ is the density of the medium at point x. This equation is solved by marching backward in energy, i.e. we prescribe $\psi^0(\epsilon_{\max},x)=0$ and $\psi^1(\epsilon_{\max},x)=0_{\mathbb{R}^3}$ at initial energy $\epsilon_{\max}$ (which means that electrons have bounded energy) and we solve (1.1) from $\epsilon_{\max}$ to 0. This choice is motivated by two reasons. First, the system (1.1) is obtained from the following kinetic equation [11]

$$\frac{\Omega}{\rho(x)}.\nabla_x\psi(x,\epsilon,\Omega)=\partial_\epsilon(S(\epsilon)\psi)(x,\epsilon,\Omega)+T(\epsilon)\partial_\mu\left((1-\mu^2)\partial_\mu\psi\right)(x,\epsilon,\Omega), \tag{1.2}$$

by extracting moments (integrating over all $\Omega=(\mu,\sqrt{1-\mu^2}\cos\phi,\sqrt{1-\mu^2}\sin\phi)\in S^2$ gives (1.1a) and multiplying (1.2) by Ω and integrating over all $\Omega\in S^2$ gives (1.1b)). One realizes that the collision operator in (1.2) is backward parabolic in ϵ. Indeed it is ill-posed when working in the direction of increasing ϵ. Second, this choice is also consistent with the physics. Indeed the electrons only loses electrons in the medium. They are injected with a maximum energy which progressively decreases. In order to be consistent with both the underlying kinetic equation and the physics behind it, we always solve (1.1) from a maximum energy $\epsilon_{\max}$ to 0.

1.1 M_1 model

The system (1.1) is composed of 4 equations with 9 unknowns (scalar ψ^0, vector ψ^1 and symmetric matrix with known trace ψ^2). It is closed using the entropy minimization principle [19]:

We seek the function $\psi_M\geq0$ minimizing the Boltzmann entropy function

$$\mathcal{H}(f)=\int_{S^2}f(\Omega)\ln f(\Omega)d\Omega$$

under the constraint of realizing the moments of order 0 and 1, i.e.

$$\int_{S^2}f(\Omega)d\Omega=\psi^0,\quad \int_{S^2}\Omega f(\Omega)d\Omega=\psi^1.$$

We close the system (1.1) *by fixing ψ^2 as the 2nd order moment of ψ_M*

$$\psi^2=\int_{S^2}\Omega\Omega^T\psi_M(\Omega). \tag{1.3}$$