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Geometric Numerical Integration for Peakon b-Family Equations

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Abstract. In this paper, we study the Camassa-Holm equation and the Degasperis-Procesi equation. The two equations are in the family of integrable peakon equations, and both have very rich geometric properties. Based on these geometric structures, we construct the geometric numerical integrators for simulating their soliton solutions. The Camassa-Holm equation and the Degasperis-Procesi equation have many common properties, however they also have the significant difference, for example there exist the shock wave solutions for the Degasperis-Procesi equation. By using the symplectic Fourier pseudo-spectral integrator, we simulate the peakon solutions of the two equations. To illustrate the smooth solitons and shock wave solutions of the DP equation, we use the splitting technique and combine the composition methods. In the numerical experiments, comparisons of these two kinds of methods are presented in terms of accuracy, computational cost and invariants preservation.

AMS subject classifications: 35L65, 65M70, 65N06, 65P10, 74J40

Key words: Symplectic integrator, splitting method, WENO scheme, multisymplectic integrator, peakon, shockpeakon.

1 Introduction

In the soliton theory, it is important to study the completely integrable nonlinear partial differential equations which arise from approximating the shallow water systems. Such equations usually have infinite number of conservation laws, and admit the soliton solutions which have the localized spatial structures and show the particle-like scattering

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behavior. In this paper, our interest is to study a family of third-order dispersive nonlinear equations

$$u_t+c_0u_x-u_{xxt}+(\beta+1)uu_x-\beta u_xu_{xx}-uu_{xxx}=0, \tag{1.1}$$

where β is a bifurcation or balance parameter which provides a balance for the behavior of nonlinear solutions, and uu_x refers to the advection term which coefficient $\beta+1$ often causes a steepening of wave. The family of PDEs (1.1) includes two important equations: the Camassa-Holm (CH) equation (when $\beta=2,c_0=2\kappa^2$) [5] and the Degasperis-Procesi (DP) equation (when $\beta=3,c_0=3\kappa^3$) [14]. Both the CH equation and the DP equation can be viewed as the model equations of shallow water waves [5,6,12,22]. By presenting the Lax pair and bi-Hamiltonian structure, it has been proved that the two systems are completely integrable [6,13]. In the absence of linear dispersion term u_{xxx}, with the nonlinear dispersion term uu_{xxx} the DP and CH equations have the novel properties one of which is that they admit the peakon soliton solutions [5, 13, 14]. To derive the peakon soliton solutions exactly, we can use the inverse scattering techniques [1,2,28,29].

Although the CH and the DP equations have shared some common properties, they are divergent by having the major differences: Lax pair equation and wave breaking phenomena [13,27]. The isospectral problem in Lax pair for the DP equation is the third-order equation while one is the second-order equation for the CH equation. Thus, the DP equation has more types of solutions than the CH equation. It is illustrated in [9,27] that the DP equation has not only the peakon solutions, but also the shockpeakon solutions which produce the difficulty to capture the shock wave numerically. Compared with the CH equation, there exist only few numerical methods and corresponding numerical analysis for the DP equation. The existing numerical methods for the DP equation include the operator splitting schemes [8], the particle method based on the multi-shockpeakon solutions [20], the conservative finite difference schemes [32], local discontinuous Galerkin (LDG) methods [41], the direct discontinuous Galerkin (DDG) method [25], the compact finite difference method [44] and the spectral method [40], etc.

As a class of conservative PDEs, the DP and CH equations have many conservative properties, such as the bi-Hamiltonian structures, infinite number of conservation laws etc. A natural idea of numerical computation for the two systems is to construct the numerical methods which can carry as much as possible these intrinsic properties. Geometric numerical integrators are a kind of numerical methods constructed based on this idea [19]. Compared with the non-geometric numerical integrators, geometric numerical integrators usually illustrate the remarkable capacity to capture the dynamical behavior of the given system over long time [19]. Based on Hamiltonian structure of the DP and CH equations, our purpose of this paper is to construct the corresponding numerical methods. To simulate the peakon solutions of the two equations, we present the numerical discretization which is given by using the pseudo-spectral method in space and symplectic integrator in time. By means of fast Fourier transform (FFT), the numerical solution with exponential convergency can be obtained if the solution is smooth enough. The numerical results show the long-term stability of the symplectic pseudo-