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REVIEW ARTICLE

Upwind and High-Resolution Methods for Compressible Flow: From Donor Cell to Residual-Distribution Schemes†

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Abstract. In this paper I review three key topics in CFD that have kept researchers busy for half a century. First, the concept of upwind differencing, evident for 1-D linear advection. Second, its implementation for nonlinear systems in the form of high-resolution schemes, now regarded as classical. Third, its genuinely multidimensional implementation in the form of residual-distribution schemes, the most recent addition. This lecture focuses on historical developments; it is not intended as a technical review of methods, hence the lack of formulas and absence of figures.

Key words: Upwind difference; high-resolution method; compressible flow; nonlinear conservation laws.

1 Why upwind differencing?

Upwind differencing is a way of differencing the spatial-derivative terms in the advection equation, and is almost as old as CFD, starting with the work of Courant, Isaacson and Rees (1952 [12]). In their paper, the choice of an upwind-biased stencil follows rather

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naturally from the "backward" variant of the Method of Characteristics. In the course of the decades further evidence has been gathered in support of upwind disceretizations.

- *Godunov, 1959.* The Russian mathematician S. K. Godunov [18] favored the first-order-accurate upwind scheme among a family of simple descretizations, because it is the most accurate one that preserves the monotonicity of an initially monotone discrete solution.
- *Fromm, 1968.* IBM researcher Jacob Fromm [16] constructed higher-order advection schemes with low dispersive error, by combining schemes with predominantly negative and predominantly positive phase errors: "Zero Average Phase Error Method." The resulting schemes turn out to be upwind biased.
- *Wesseling, 1973.* Dutch aerospace engineer (turned numerical analyst) Pieter Wesseling [64] used Parseval's theorem to relate the numerical error committed by advection schemes to the Fourier transform of the intial-value distribution. For two different families of advection schemes he found it is an upwind scheme that minimizes the L_2-error made in one time step *if the initial values contain a discontinuity.* This would indicate upwind schemes may be the preferred choice for compressible flow, where shock discontinuities are common and arise even from the smoothest initial data.
- *van Leer, 1986.* Reversing Fromm's procedure, Dutch astrophysicist (turned aerospace engineer) Bram van Leer [35] developed an operational definition of upwind schemes. For example, a linear update scheme of the form

  $$u^j = \sum_k C_k(\nu) u_{j+k}, \tag{1.1}$$

  is called *upwind-biased with respect to the CFL-number range (0,1)* if and only if its coefficents satisfy the symmetry relation

  $$C_k(1-\nu) = C_{-k-1}(\nu); \tag{1.2}$$

  here ν is the CFL number. For any such scheme, the result of one step with CFL number ν followed by a step with CFL-number $1-\nu$, is free of dispersion, a quantitative expression of Fromm's idea. It further follows that such a scheme is free of dispersion for $\nu = 1/2$.
- *Jeltsch, 1987.* Mathematicians including Swiss Rolf Jeltsch [30], searching for advection stencils with the greatest potential accuracy for a given number of grid-points, have proved that these stencils are upwind biased.

The price one has to pay for all this goodness is the computational effort in determining the advection direction. That is trivial for a linear 1-D advection equation, but a major effort for nonlinear advection operators hidden in nonlinear systems of multidimensional conservation laws.