REVIEW ARTICLE

Upwind and High-Resolution Methods for Compressible Flow: From Donor Cell to Residual-Distribution Schemes†

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Abstract. In this paper I review three key topics in CFD that have kept researchers busy for half a century. First, the concept of upwind differencing, evident for 1-D linear advection. Second, its implementation for nonlinear systems in the form of high-resolution schemes, now regarded as classical. Third, its genuinely multidimensional implementation in the form of residual-distribution schemes, the most recent addition. This lecture focuses on historical developments; it is not intended as a technical review of methods, hence the lack of formulas and absence of figures.

Key words: Upwind difference; high-resolution method; compressible flow; nonlinear conservation laws.

1 Why upwind differencing?

Upwind differencing is a way of differencing the spatial-derivative terms in the advection equation, and is almost as old as CFD, starting with the work of Courant, Isaacson and Rees (1952 [12]). In their paper, the choice of an upwind-biased stencil follows rather
naturally from the “backward” variant of the Method of Characteristics. In the course of
the decades further evidence has been gathered in support of upwind discretizations.

- **Godunov, 1959.** The Russian mathematician S. K. Godunov [18] favored the first-
  order-accurate upwind scheme among a family of simple discretizations, because it
  is the most accurate one that preserves the monotonicity of an initially monotone
  discrete solution.

- **Fromm, 1968.** IBM researcher Jacob Fromm [16] constructed higher-order advection
  schemes with low dispersive error, by combining schemes with predominantly nega-
  tive and predominantly positive phase errors: “Zero Average Phase Error Method.”
  The resulting schemes turn out to be upwind biased.

- **Wesseling, 1973.** Dutch aerospace engineer (turned numerical analyst) Pieter Wes-
  seling [64] used Parseval’s theorem to relate the numerical error committed by ad-
  vention schemes to the Fourier transform of the initial-value distribution. For two
  different families of advection schemes he found it is an upwind scheme that mini-
  mizes the $L_2$-error made in one time step if the initial values contain a discontinuity.
  This would indicate upwind schemes may be the preferred choice for compressible
  flow, where shock discontinuities are common and arise even from the smoothest
  initial data.

- **van Leer, 1986.** Reversing Fromm’s procedure, Dutch astrophysicist (turned aerospace
  engineer) Bram van Leer [35] developed an operational definition of upwind schemes.
  For example, a linear update scheme of the form

  \[ u^j = \sum_k C_k(\nu)u_{j+k}, \]  

  (1.1)

  is called upwind-biased with respect to the CFL-number range $(0,1)$ if and only if its
  coefficients satisfy the symmetry relation

  \[ C_k(1 - \nu) = C_{-k-1}(\nu); \]  

  (1.2)

  here $\nu$ is the CFL number. For any such scheme, the result of one step with CFL
  number $\nu$ followed by a step with CFL-number $1 - \nu$, is free of dispersion, a quan-
  titative expression of Fromm’s idea. It further follows that such a scheme is free of
  dispersion for $\nu = 1/2$.

- **Jeltsch, 1987.** Mathematicians including Swiss Rolf Jeltsch [30], searching for advec-
  tion stencils with the greatest potential accuracy for a given number of grid-points,
  have proved that these stencils are upwind biased.

The price one has to pay for all this goodness is the computational effort in determining
the advection direction. That is trivial for a linear 1-D advection equation, but a major
effort for nonlinear advection operators hidden in nonlinear systems of multidimensional
conservation laws.