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Construction and Analysis of an Adapted Spectral Finite Element Method to Convective Acoustic Equations

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Abstract. The paper addresses the construction of a non spurious mixed spectral finite element (FE) method to problems in the field of computational aeroacoustics. Based on a computational scheme for the conservation equations of linear acoustics, the extension towards convected wave propagation is investigated. In aeroacoustic applications, the mean flow effects can have a significant impact on the generated sound field even for smaller Mach numbers. For those convective terms, the initial spectral FE discretization leads to non-physical, spurious solutions. Therefore, a regularization procedure is proposed and qualitatively investigated by means of discrete eigenvalues analysis of the discrete operator in space. A study of convergence and an application of the proposed scheme to simulate the flow induced sound generation in the process of human phonation underlines stability and validity.

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1 Introduction

Constructing a numerical scheme to compute the acoustic field in flowing media is not an easy task. The first step is the choice of an appropriate physical model. The simplest one towards aeroacoustics is Lighthill's analogy [1], whose main drawback is that

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it does not separate flow and acoustic quantities and so the computed fluctuating pressure just converges to the acoustic one for observation points far away from the turbulent source region, where the mean flow velocity becomes zero. For acoustic propagation in the presence of a (non-uniform) mean flow or compressible, non-isentropic media, formulations based on the linearized Euler equations (LEE) [2] and acoustic perturbation equations [3–5] seem to be the most realistic models. From the mathematical point of view, these models present an important problem, because the involved differential operators do not correspond to any classical functional space, and therefore no classical finite element fit to its numerical solution. Actually, the use of classical finite element approximations (in H^1 or $H(\mathbf{div})$) generates substantial parasitic waves produced by numerical spurious modes. The discontinuous Galerkin methods seem well adapted for the approximation of linearized Euler equations since they do not require any regularity of the solution. On the other hand, these methods are able to suppress parasitic waves by the use of a dissipative jump [6]. First implemented for triangular and tetrahedral meshes for these equations [7–9], these methods were then adapted to quadrilateral and hexahedral meshes using mass-lumped spectral elements [10]. Another way to get a spurious free solution for a continuous finite element approximation is to get a coercive operator by adding a differential penalty term. This method, introduced by [11] for Maxwell's equations, was applied to the time-harmonic Galbrun's equations [12] (which are equivalent to LEE) to which a penalty term using a **curl**-operator was added to get coercivity. However, this approach implies the numerical treatment of an additional operator, which substantially increases the computational time, in particular in time domain. In this paper, we present a finite element approximation in which the acoustic pressure is sought in H^1 and the irrotational part of the fluctuating velocity (acoustic particle velocity) in $(L^2)^d$, $d=2,3$. The discontinuous character of the second variable enables us to add a jump term which, as for Maxwell's equations [13], gets rid of spurious modes. This approach leads to a method based on continuous finite elements which is spurious free for a small additional cost.

After a brief presentation in Section 2 of the acoustic perturbation equations we present in Section 3 a straightforward extension of the mixed spectral FE to these equations. We illustrate and explain why such straightforward derivation is unstable. In Section 4 we construct and study by a discrete eigenvalue analysis a stabilization and penalization procedures to rectify the previously proposed formulation. Finally in Section 5 we present a numerical convergence analysis and a successful extension of the proposed scheme to three dimensional setups and spatially varying mean flows by means of a numerical computation of the aeroacoustic sound generated in the human phonation process.

2 Acoustic Perturbation Equations

A known system of perturbation equations for aeroacoustic computations based upon incompressible flow data is APE-1 as proposed by R. Ewert, which is a system of two PDEs (Eq. (47)-(48) in [3]). Assuming constant temperature and small Mach number, in