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A Posteriori Error Estimates for Conservative Local Discontinuous Galerkin Methods for the Generalized Korteweg-de Vries Equation

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Abstract. We construct and analyze conservative local discontinuous Galerkin (LDG) methods for the Generalized Korteweg-de-Vries equation. LDG methods are designed by writing the equation as a system and performing separate approximations to the spatial derivatives. The main focus is on the development of conservative methods which can preserve discrete versions of the first two invariants of the continuous solution, and a posteriori error estimates for a fully discrete approximation that is based on the idea of dispersive reconstruction. Numerical experiments are provided to verify the theoretical estimates.

AMS subject classifications: 65M12, 65M60, 35Q53

Key words: Discontinuous Galerkin methods, Korteweg-de-Vries equation, a posteriori error estimate, conservative methods.

1 Introduction

In this paper we consider the Generalized Korteweg-de Vries (GKdV) equation posed with periodic boundary conditions

$$\begin{cases} u_t+(u^{p+1})_x+\epsilon u_{xxx}=0, & 0<x<1, \ t>0, \\ u(x,0)=u^0(x), & 0<x<1, \end{cases} \tag{1.1}$$

where p is a positive integer and ϵ is a positive parameter. The GKdV equation belongs to a class of equations featuring nonlinear and dispersive effects that are widely used to model the propagation of physical waves.

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Since the discovery of the solitons in the sixties there has been intense interest and resulting research activity on the well-posedness as well as the numerical treatment of (1.1) and other nonlinear dispersive equations. The problem (1.1) is locally well-posed in a wide range of function classes, but it is also known that solutions do not exist for all time and singularity formation may occur, as can be gleaned from [3, 23, 24]. In parallel to the analytical developments, intense attention focused on developing methods for the numerical treatment of (1.1) resulting in schemes belonging to all the known classes of numerical methods including finite difference, finite element, finite volume and spectral methods as well as "special" methods based on the inverse scattering transform. We refer to [9] and the references therein for a survey of such works. However, it must be said that a combination of the nonlinearity and the dispersive term u_{xxx} (which is a derivative of odd order) makes the rigorous treatment of issues such as stability and convergence quite difficult. Whereas a few early works contained such rigorous treatments, the work of Shu and coworkers in the new century on discontinuous Galerkin (DG) methods constituted an important development through the construction of a dissipative dispersive projection operator [11, 27]. In [9] two of the authors advanced the paradigm and showed that a conservative version of the dissipative operator constructed in [11, 27] has beneficial numerical properties such as slower growth of the errors over long time intervals.

As in [9], the numerical methods discussed here are the DG methods. They are characterized by the use of piecewise polynomial spaces that are totally discontinuous, and were originally devised to solve hyperbolic conservation laws with only first order spatial derivatives, e.g. [13–15, 17, 18, 25]. They allow arbitrarily unstructured meshes, and have a compact stencil; moreover, they easily accommodate arbitrary h-p adaptivity. The DG methods were later generalized to the local DG (LDG) methods by Cockburn and Shu to solve the convection-diffusion equation [16], motivated by successful numerical experiments from Bassi and Rebay for the compressible Navier-Stokes equations [6]. As a result, the LDG methods have been applied to solve various partial differential equations (PDEs) containing higher-order derivatives. We refer to the review paper [26] for more details. The LDG method, in contrast to the so-called primitive variable formulations, is characterized by writing the evolution equation as a system by considering each spatial derivative as a dependent variable, one benefit of such an approach being the simultaneous approximation of the spatial derivatives. For the KdV-type equations (1.1), an LDG method was first developed in [29], in which a sub-optimal error estimate was provided for the linearized problem. In [27], Xu and Shu proved the $k+1/2$-th order convergence rate for the LDG method applied to the fully nonlinear KdV equation. Later, an optimal L^2 error estimate was derived in [28] for the linearized equation. Recently, there has been a different approach in solving the KdV equations by using the DG method directly without introducing any auxiliary variables nor rewriting the original equation into a larger system. Cheng and Shu proposed such DG methods in [12] for PDEs involving high-order derivatives, and an energy-conserving DG method for the KdV equation was developed by Bona et al. in [9]. The superconvergence property of the LDG methods for the linearized KdV equation has been studied in [20].