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Optimal Superconvergence of Energy Conserving Local Discontinuous Galerkin Methods for Wave Equations

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Abstract. This paper is concerned with numerical solutions of the LDG method for 1D wave equations. Superconvergence and energy conserving properties have been studied. We first study the superconvergence phenomenon for linear problems when alternating fluxes are used. We prove that, under some proper initial discretization, the numerical trace of the LDG approximation at nodes, as well as the cell average, converge with an order $2k+1$. In addition, we establish $k+2$-th order and $k+1$-th order superconvergence rates for the function value error and the derivative error at Radau points, respectively. As a byproduct, we prove that the LDG solution is superconvergent with an order $k+2$ towards the Radau projection of the exact solution. Numerical experiments demonstrate that in most cases, our error estimates are optimal, i.e., the error bounds are sharp. In the second part, we propose a fully discrete numerical scheme that conserves the discrete energy. Due to the energy conserving property, after long time integration, our method still stays accurate when applied to nonlinear Klein-Gordon and Sine-Gordon equations.

AMS subject classifications: 65L20, 65M12, 65N12

Key words: Local discontinuous Galerkin methods (LDG), wave equations, superconvergence, energy conserving.

1 Introduction

We study the local discontinuous Galerkin (LDG) method for the following 1D wave equations

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$$\begin{aligned} &u_{tt}=u_{xx}+f(u), \qquad (x,t)\in[a,b]\times[0,T],\\ &u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x) \end{aligned} \tag{1.1}$$

with the periodic boundary condition. We investigate superconvergence property of the LDG method for the linear problem, and an energy conserving fully discrete scheme for the nonlinear problem.

The superconvergence of DG, especially LDG methods has been one of the hot research spots in recent years. We refer to [1, 2, 7, 16, 22, 24, 28] for the investigation related to ordinary and delay differential equations, [3, 8, 11, 26] for hyperbolic equations and [9, 12, 21, 27] for convection-diffusion equations. Superconvergence of the LDG method for wave equations has also been investigated. In particular, Baccouch proved a local superconvergence rate of $k+2$ and a global superconvergence rate of $k+3/2$ at Radau points for 1D wave equations [5] and [6]. Xing et al. presented an energy conserving LDG method for wave propagation and also proved a $k+3/2$-th superconvergence rate of the LDG approximation to a special projection of the exact solution [25]. However, numerical experiments for wave equations indicated that aforementioned theoretical rates were not sharp. In order to establish the optimal superconvergence results for LDG methods, some new analysis tools were developed very recently. The main idea was to construct, in the discrete space, a special function, which was used to correct the error between the LDG solution and the Gauss-Radau projection of the exact solution. Thanks to the correction function, we established the optimal superconvergence rate of the LDG method for the hyperbolic problems [8] and parabolic problems [9], respectively. Now, we try to complete the jigsaw puzzle and continue the study of the optimal superconvergence results for wave equations.

The first part of this paper is to revisit superconvergence properties of the LDG method for wave equations. We provide a rigorous mathematical proof of the optimal superconvergence rate for the LDG method. On one hand, for the first time we show that the numerical trace of the LDG approximation at nodes, as well as the cell average, is superconvergent with the order $2k+1$. On the other hand, the function value approximation and the derivative approximation are superconvergent with orders $k+2$ and $k+1$ at Radau points, respectively. The superconverge results at the Radau points improve these in [6] and [25]. As a byproduct, a $k+2$-th superconvergence order of the LDG solution towards a particular Gauss-Radau projection of the exact solution is obtained. Moreover, our analysis also leads to some interesting numerical discoveries, which will be reported in the numerical experiments.

The analysis of this paper is based on the correction function idea, which is motivated from its successful application to hyperbolic [8] and parabolic equations [9]. But the construction of the correct functions for wave equations is more technical and complicated. It is different from the DG method for hyperbolic equations [8] due to the interplay between two correction functions. We have to rewrite the wave equation as a system of first order partial differential equations with respect to the spacial variable x. The correction