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Construction of Symplectic Runge-Kutta Methods for Stochastic Hamiltonian Systems

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Abstract. We study the construction of symplectic Runge-Kutta methods for stochastic Hamiltonian systems (SHS). Three types of systems, SHS with multiplicative noise, special separable Hamiltonians and multiple additive noise, respectively, are considered in this paper. Stochastic Runge-Kutta (SRK) methods for these systems are investigated, and the corresponding conditions for SRK methods to preserve the symplectic property are given. Based on the weak/strong order and symplectic conditions, some effective schemes are derived. In particular, using the algebraic computation, we obtained two classes of high weak order symplectic Runge-Kutta methods for SHS with a single multiplicative noise, and two classes of high strong order symplectic Runge-Kutta methods for SHS with multiple multiplicative and additive noise, respectively. The numerical case studies confirm that the symplectic methods are efficient computational tools for long-term simulations.

AMS subject classifications: 65C30, 60H35, 65P10

Key words: Stochastic differential equation, Stochastic Hamiltonian system, symplectic integration, Runge-Kutta method, order condition.

1 Introduction

Consider the following Cauchy problem for stochastic differential equations (SDEs):

$$dX_t = a(t,X_t)dt + \sum_{k=1}^{m} b_k(t,X_t) * dw_t^k, \qquad X_{t_0} = x_0, \tag{1.1}$$

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where $X,a(t,x^1,\cdots,x^r)$, $b_k(t,x^1,\cdots,x^r)$ are r-dimensional column-vectors with the components X^i,a^i,b^i_j, $i=1,\cdots,r$, $a,b_k\in C^{2\eta}(\mathbb{R}^r,\mathbb{R}^r)$, $\eta=1,2,\cdots$, and where w^k_t, $k=1,\cdots,m$, are independent standard Wiener processes. We write $*dw^k_t=dw^k_t$ in the case of an Itô stochastic integral and $*dw^k_t=\circ dw^k_t$ for a Stratonovich stochastic integral.

Let us write a system of SDEs of even dimension $r=2d$ in the form of stochastic Hamiltonian systems (SHS) in the sense of Stratonovich:

$$\begin{aligned} dP^i &= -\frac{\partial H_0(t,P,Q)}{\partial Q^i}dt - \sum_{k=1}^{m}\frac{\partial H_k(t,P,Q)}{\partial Q^i}\circ dw^k_t, \qquad P(t_0)=p, \\ dQ^i &= \frac{\partial H_0(t,P,Q)}{\partial P^i}dt + \sum_{k=1}^{m}\frac{\partial H_k(t,P,Q)}{\partial P^i}\circ dw^k_t, \qquad Q(t_0)=q \end{aligned} \tag{1.2}$$

for $d,m\geq 1$ with an m-dimensional Wiener process $(w_t)_{t\geq 0}$ and $t\in\mathbb{R}$, where P,Q,p,q are d-dimensional vectors with components P^i,Q^i,p^i,q^i, $i=1,2,\cdots,d$. The SHS (1.2) includes both Hamiltonian systems with additive or multiplicative noise.

For SHS (1.2), [28] established the theory about the stochastic symplectic methods which preserve the symplectic structure of the SDEs. Tretyakov and Tret'jakov [40] considered numerical methods for Hamiltonian systems with external noise. Seesselberg et al. [38] investigated the numerical simulation of singly noisy Hamiltonian systems and their application to particle storage rings. Misawa [29] proposed an energy conservative stochastic difference scheme for a one-dimensional stochastic Hamilton dynamical system. Milstein, Repin and Tretyakov [25,26] investigated symplectic integration of SHS (1.2) with additive and multiplicative noise, respectively. Hong, Scherer and Wang [14,15] investigated numerical methods for linear stochastic oscillator with additive noise. Milstein and Tretyakov [27] presented quasi-symplectic integration for Langevin-type equations. Wang et al [41, 42] discussed variational integrators and generating functions of SHS (1.2). Deng, Anton and Wong [12] proposed some high order symplectic schemes based on generating functions. Abdulle, Cohen, Vilmart and Zygalakis [1] proposed a new methodology for constructing numerical integrators with high weak order for the time integration of stochastic differential equations based on modified equations. Hong, Zhai and Zhang [17] proposed discrete gradient approach to stochastic differential equations with a conserved quantity. Cohen and Duardin [8] proposed a new class of energy-preserving numerical schemes for stochastic Hamiltonian systems with noncanonical structure matrix in the Stratonovich sense. Hong, Xu and Wang [16] investigated quadratic invariant-preserving SRK methods for SDEs possessing an invariant in the sense of Stratonovich. Recently, Cristina, Deng and Wong [9, 10] discussed symplectic schemes for SHS and stochastic systems preserving Hamiltonian functions, respectively. Using generating functions, Wang [42] presented the generalization of a symplectic Runge-Kutta method for SHS with a single noise in the sense of Stratonovich. Ma, Ding and Ding [23] presented the symplectic conditions of SRK methods for SHS with a single noise in the sense of Stratonovich. And the above two works are concerned about the strong convergence case. Here we will discuss the more general cases that include the weak convergence case, the multiple noise case and the Itô case. An attempt to construct practical SRK methods preserving the symplectic property for various types of SHS, introduced in this paper.

The outline of the rest of this paper is as follows. In Section 2 we consider the symplecticity of stochastic Runge-Kutta (SRK) methods with weak or strong order for SHS with multiplicative noise. We also discuss the symplecticity of SRK methods with weak or strong order for SHS with special separable Hamiltonians and for SHS with multiple additive noise in Section 3 and Section 4, respectively. In Section 5, some numerical experiments are carried out in order to justify our theoretical results.

2 The symplecticity of SRK methods for SHS with multiplicative noise

In this section we will study the symplecticity of SRK methods that are used to solve SHS with multiplicative noise. For this purpose, we will be concerned with a uniform partition on $\mathcal{L}\subset\mathbb{R}$ with nodal points $t_n=t_0+nh$, $n=0,1,\cdots,N$, where $h=(T-t_0)/N$, $N=1,2,\cdots$ is the stepsize. Numerical schemes for SDEs are recursive methods where trajectories of the solution are computed at discrete time steps. We first recall the concepts of convergence for the numerical integration of SDEs.

Definition 2.1. A discrete time approximation Y_N^h is said to be convergent with a strong order κ (respectively, weak order of ν) to solution of SDE at time τ if there exists a constant C such that

$$E(|Y_N^h-X(\tau)|)<Ch^{\kappa}\ \text{(strong)},\quad |E(\varphi(Y_N^h))-E(\varphi(X(\tau)))|<Ch^{\nu}\ \text{(weak)}$$

for any fixed $\tau=nh\in\mathcal{L}$ and h sufficiently small and for all functions $\varphi:\mathbb{R}^d\to\mathbb{R}\in C_P^{2(\nu+1)}$. Here $C_P^{2(\nu+1)}$ denotes the space of $2(\nu+1)$ times continuously differentiable functions $\mathbb{R}^d\to\mathbb{R}$ with all partial derivatives with polynomial growth.

In differential geometry, the exterior product $df\wedge dg$ of the functions $f,g:\mathbb{R}^{2d}\to\mathbb{R}$ on $\phi,\psi\in\mathbb{R}^{2d}$ is given by $df\wedge dg(\phi,\psi)=df(\phi)dg(\psi)-df(\psi)dg(\phi)$, and represents the oriented area of the image of the parallelogram with sides $df(\phi)$ and $dg(\psi)$ on the $df(\phi),dg(\psi)$-plane. The stochastic flow $(p,q)\to(P,Q)$ of the SHS (1.2) preserves the symplectic structure (Theorem 2.1 of [26]) as follows:

$$dP\wedge dQ=dp\wedge dq, \tag{2.1}$$

i.e., the sum of the oriented areas of projections of a two-dimensional surface onto the coordinate planes (p^i,q^i) is invariant. Consider the differential two-form

$$dp\wedge dq=dp^1\wedge dq^1+\cdots+dp^d\wedge dq^d.$$

To avoid confusion, we note that the differentials in (1.2) and (2.1) have different meanings. In (1.2), P,Q are treated as functions of time, and p,q are fixed parameters, while differentiation in (2.1) is made with respect to the initial data p,q. We say that a numerical method based on a one step approximation $P_{n+1}=P_{n+1}(t_n+h;t_n,P_n,Q_n)$, $Q_{n+1}=Q_{n+1}(t_n+h;t_n,P_n,Q_n)$ preserves the symplectic structure if

$$dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n.$$

For convenience, we denote

$$\begin{aligned} &f^i=-\frac{\partial H_0}{\partial Q^i}, \qquad g^i=\frac{\partial H_0}{\partial P^i}, \qquad \sigma_k^i=-\frac{\partial H_k}{\partial Q^i},\\ &\gamma_k^i=\frac{\partial H_k}{\partial P^i}, \qquad i=1,\cdots,d, \qquad k=1,\cdots,m. \end{aligned} \tag{2.2}$$

2.1 Symplectic conditions of weak order SRK methods

For SDEs (1.1) in the sense of Stratonovich, a class of SRK methods with $Y_0=x_0$ is given by Rößler [35]

$$\begin{aligned} Y_{n+1}=&Y_n+\sum_{i=1}^{s}\alpha_i a\big(t_n+c_i^{(0)}h,G_i^{(0)}\big)h+\sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}b^k\big(t_n+c_i^{(1)}h,G_i^{(k)}\big)\hat{I}_k\\ &+\sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(2)}b^k\big(t_n+c_i^{(1)}h,\hat{G}^{(k)}\big)\sqrt{h} \end{aligned} \tag{2.3}$$

for $n=0,1,\cdots,N-1$ with supporting values

$$\begin{aligned} G_i^{(0)}=&Y_n+\sum_{j=1}^{s}A_{ij}^0 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0 b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)\hat{I}_l,\\ G_i^{(k)}=&Y_n+\sum_{j=1}^{s}A_{ij}^1 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{j=1}^{s}B_{ij}^1 b^k\big(t_n+c_j^{(1)}h,G_j^{(k)}\big)\hat{I}_k\\ &+\sum_{\substack{l=1\\ l\neq k}}^{m}\sum_{j=1}^{s}B_{ij}^3 b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)\hat{I}_l,\\ \hat{G}_i^{(k)}=&Y_n+\sum_{j=1}^{s}A_{ij}^2 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{l=1,l\neq k}^{m}\sum_{j=1}^{s}B_{ij}^2 b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)\frac{\hat{I}_{kl}}{\sqrt{h}} \end{aligned}$$

for $i=1,\cdots,s$ and $k=1,\cdots,m$. The random variables are defined by

$$\hat{I}_{kl}=\begin{cases} \hat{I}_k\tilde{I}_l & \text{if} \quad l<k,\\ -\hat{I}_l\tilde{I}_k & \text{if} \quad k<l \end{cases}$$

with independent random variables $\hat{I}_k$, $1\le k\le m$, possessing the moments

$$E(\hat{I}_k^q)=\begin{cases} 0 & \text{for} \quad q\in\{1,3,5\}, \\ (q-1)h^{q/2} & \text{for} \quad q\in\{2,4\}, \\ \mathcal{O}(h^{q/2}) & \text{for} \quad q\ge 6, \end{cases} \tag{2.4}$$

and $\tilde{I}_k$, $1\le k\le m-1$, having the moments

$$E(\tilde{I}_k^q)=\begin{cases} 0 & \text{for} \quad q\in\{1,3\}, \\ h & \text{for} \quad q=2, \\ \mathcal{O}(h^{q/2}) & \text{for} \quad q\ge 4. \end{cases} \tag{2.5}$$

We choose $\hat{I}_k$ as three point distributed random variables with $P(\hat{I}_k=\pm\sqrt{3h})=\frac{1}{6}$ and $P(\hat{I}_k=0)=\frac{2}{3}$, and $\tilde{I}_k$ as two point distributed random variables with $P(\tilde{I}_k=\pm\sqrt{h})=\frac{1}{2}$. Applying SRK methods (2.3) with $\beta_i^{(2)}=0$, $i=1,\cdots,s$ to SHS (1.2), we obtain

$$\begin{aligned} P_{n+1}&=P_n+h\sum_{i=1}^{s}\alpha_i f(t_n+c_i^{(0)}h,p_i^{(0)},q_i^{(0)})+\sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\sigma^k\big(t_n+c_i^{(1)}h,p_i^{(k)},q_i^{(k)}\big)\hat{I}_k, \\ Q_{n+1}&=Q_n+h\sum_{i=1}^{s}\alpha_i g(t_n+c_i^{(0)}h,p_i^{(0)},q_i^{(0)})+\sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\gamma^k\big(t_n+c_i^{(1)}h,p_i^{(k)},q_i^{(k)}\big)\hat{I}_k \end{aligned} \tag{2.6}$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$\begin{aligned} p_i^{(0)}&=P_n+h\sum_{j=1}^{s}A_{ij}^0 f(t_n+c_j^{(0)}h,p_j^{(0)},q_j^{(0)})+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0\sigma^l\big(t_n+c_i^{(1)}h,p_j^{(l)},q_j^{(l)}\big)\hat{I}_l, \\ q_i^{(0)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^0 g(t_n+c_j^{(0)}h,p_j^{(0)},q_j^{(0)})+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0\gamma^l\big(t_n+c_i^{(1)}h,p_j^{(l)},q_j^{(l)}\big)\hat{I}_l, \\ p_i^{(k)}&=P_n+h\sum_{j=1}^{s}A_{ij}^1 f(t_n+c_j^{(0)}h,p_j^{(0)},q_j^{(0)})+\sum_{j=1}^{s}B_{ij}^1\sigma^l\big(t_n+c_j^{(1)}h,p_j^{(k)},q_j^{(k)}\big)\hat{I}_k \\ &\quad+\sum_{l=1,l\neq k}^{m}\sum_{j=1}^{s}B_{ij}^3\sigma^l\big(t_n+c_j^{(1)}h,p_j^{(l)},q_j^{(l)}\big)\hat{I}_l, \\ q_i^{(k)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^1 g(t_n+c_j^{(0)}h,p_j^{(0)},q_j^{(0)})+\sum_{j=1}^{s}B_{ij}^1\gamma^l\big(t_n+c_j^{(1)}h,p_j^{(k)},q_j^{(k)}\big)\hat{I}_k \\ &\quad+\sum_{l=1,l\neq k}^{m}\sum_{j=1}^{s}B_{ij}^3\gamma^l\big(t_n+c_j^{(1)}h,p_j^{(l)},q_j^{(l)}\big)\hat{I}_l. \end{aligned} \tag{2.7}$$

These SRK methods can be characterized by the tableau

$$\begin{array}{c|c|c|c} c^{(0)} & A^0 & B^0 & \\ \hline c^{(1)} & A^1 & B^1 & B^3 \\ \hline & \alpha^T & {\beta^{(1)}}^T & \end{array}$$

For SRK methods (2.6) and (2.7), we can obtain the following theorem.

Theorem 2.1. *For SHS* (1.2) *and* (2.1)*, if the coefficients of SRK methods* (2.6) *and* (2.7) *satisfy*

$$\alpha_i\alpha_j - \alpha_j A^0_{ji} - \alpha_i A^0_{ij} = 0, \tag{2.8}$$

$$\alpha_i\beta^{(1)}_j - \alpha_i B^0_{ij} - \beta^{(1)}_j A^1_{ji} = 0, \tag{2.9}$$

$$\beta^{(1)}_i\beta^{(1)}_j - \beta^{(1)}_j B^1_{ji} - \beta^{(1)}_i B^1_{ij} = 0, \tag{2.10}$$

$$\beta^{(1)}_i\beta^{(1)}_j - \beta^{(1)}_j B^3_{ji} - \beta^{(1)}_i B^3_{ij} = 0 \tag{2.11}$$

for all $i,j=1,\cdots,s$, then it preserve symplectic structure, i.e., $dP_{n+1}\wedge dQ_{n+1} = dP_n\wedge dQ_n$.

Proof. See Appendix. □

Remark 2.1. If we apply deterministic Runge-Kutta methods (2.6) with $B^0_{ij}=0$, $B^1_{ij}=0$, $\beta^{(1)}_i = 0$ $(i,j=1,\cdots,s)$ to solve deterministic Hamiltonian systems (1.2) with $H_k(t,P,Q)=0$, $k=1,\cdots,m$, symplectic conditions (2.8)-(2.11) reduce to the symplectic conditions for deterministic Runge-Kutta methods

$$\alpha_i\alpha_j - \alpha_j A^0_{ji} - \alpha_i A^0_{ij} = 0, \quad i,j=1,\cdots,s,$$

which can be found in [13,37].

Let $e=(1,1,\cdots,1)^T\in\mathbb{R}^s$. It has been shown in [35] that SRK methods (2.6)-(2.7) will have weak global order 1.0 if

$$\alpha^T e = 1, \quad ({\beta^{(1)}}^T e)^2 = 1, \quad {\beta^{(1)}}^T B^1 e = \frac{1}{2} \tag{2.12}$$

are satisfied. Using both the order conditions (2.12) and symplectic conditions (2.8)-(2.11), we construct stochastic symplectic Runge-Kutta methods, for instance, a class of weak order 1.0 one-stage SRK methods with a tableau

$$\begin{array}{c|c|c|c} \frac{1}{2} & \frac{1}{2} & b & \\ \hline 1-b & 1-b & \frac{1}{2} & \frac{1}{2} \\ \hline & 1 & 1 & \end{array}$$

which is said SRKw1 methods, where $b\in\mathbb{R}$.

Next, let us consider the single noise case, namely $m=1$ in (1.1) and (1.2). For SDEs (1.1) in the sense of Stratonovich, a class of SRK methods is given by Rößler [31, 32], namely SRK methods (2.3) with $\beta^{(2)}_i = 0, B^3_{ij}=0$ for $i,j=1,\cdots,s$ (replace $\hat{I}_1$ by Δw_n). These SRK methods can be characterized by the tableau

$$\begin{array}{c|c|c} c^{(0)} & A^0 & B^0 \\ \hline c^{(1)} & A^1 & B^1 \\ \hline & \alpha^T & {\beta^{(1)}}^T \end{array}$$

It has been shown in [31, 32] that these SRK methods will have weak global order 2.0 if

$$
\begin{aligned}
&1.\quad \alpha^T e = 1, \qquad 2.\quad {\beta^{(1)}}^T e = 1, \qquad 3.\quad {\beta^{(1)}}^T B^1 e = \frac{1}{2},\\
&4.\quad {\beta^{(1)}}^T (B^1((B^1 e)(B^1 e))) = \frac{1}{12}, \qquad 5.\quad {\beta^{(1)}}^T A^1 e = \frac{1}{2},\\
&6.\quad \alpha^T ((B^0 e)(B^0 e)) = \frac{1}{2}, \qquad 7.\quad \alpha^T B^0 e = \frac{1}{2},\\
&8.\quad {\beta^{(1)}}^T ((A^1 e)(B^1 e)) = \frac{1}{4}, \qquad 9.\quad {\beta^{(1)}}^T (A^1 (B^1 e)) = 0,\\
&10.\quad {\beta^{(1)}}^T ((B^1 e)(B^1 e)) = \frac{1}{3}, \qquad 11.\quad {\beta^{(1)}}^T (B^1 (B^1 e)) = \frac{1}{6},\\
&12.\quad {\beta^{(1)}}^T ((B^1 e)((B^1 e)(B^1 e)) = \frac{1}{4}, \qquad 13.\quad \alpha^T A^0 e = \frac{1}{2},\\
&14.\quad {\beta^{(1)}}^T ((B^1 e)(B^1 (B^1 e))) = \frac{1}{8}, \qquad 15.\quad {\beta^{(1)}}^T (B^1 (A^1 e)) = \frac{1}{4},\\
&16.\quad {\beta^{(1)}}^T (B^1 (B^1 (B^1 e))) = \frac{1}{24}, \qquad 17.\quad \alpha^T (B^0 (B^1 e)) = \frac{1}{4}
\end{aligned}
$$

are satisfied. Using both the order conditions 1–17 and symplectic conditions (2.8)-(2.11), we construct stochastic symplectic Runge-Kutta methods, for instance, a class of weak order 2.0 four-stage symplectic SRK methods with a tableau

$$
\begin{array}{c|cccc|cccc}
\frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{5}{6}-\frac{\sqrt{3}}{3} & -\frac{1}{2} & 0 & 0\\
\frac{3}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & -\frac{1}{6}+\frac{\sqrt{3}}{3} & \frac{1}{2} & 0 & 0\\
\frac{5}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\\
\frac{7}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{6} & \frac{1}{2} & 0 & 0\\
\hline
\frac{1}{2} & -\frac{1}{6}+\frac{\sqrt{3}}{6} & \frac{1}{3}-\frac{\sqrt{3}}{6} & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b_1 & b_2 & 0 & b_3\\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0\\
\hline
 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 0
\end{array}
$$

which is said SRKw2 methods, where $b_1, b_2, b_3 \in \mathbb{R}$.

Remark 2.2. In the single noise case, the random variable $J_1 = \Delta w_n$ is used for SRK methods in [31]. As implicit methods, in order to avoid unboundedness of absolute moments of the numerical solution, we replace J_1 by $\hat{I}_k$ with $k=1$, where $\hat{I}_k$ satisfy (2.4) with $k=1$. Note that here Gaussian variable J_1 can be replaced by $\hat{I}_1$ without decreasing the weak order two of the method.

2.2 Symplectic conditions of strong order SRK methods

For SDEs (1.1) in the autonomous case and the sense of Stratonovich, a class of SRK methods with $Y_0 = x_0$ is given by Burrage and Burrage [2,6]

$$Y_{n+1} = Y_n + h\sum_{j=1}^{s}\alpha_j a(G_j) + \sum_{k=1}^{m}\sum_{j=1}^{s}\beta_j^{(k)} b^k(G_j)J_k \tag{2.13}$$

for $n = 0,1,\cdots,N-1$ with stage values

$$G_i = Y_n + h\sum_{j=1}^{s}A_{ij}a(G_j) + \sum_{k=1}^{m}\sum_{j=1}^{s}B_{ij}^k b^k(G_j)J_k,$$

where $i = 1,\cdots,s$ and Stratonovich integral $J_k = \int_{t_n}^{t_{n+1}} \circ dw_s^k$. Applying SRK methods (2.13) to SHS (1.2), we obtain

$$\begin{aligned} P_{n+1} &= P_n + h\sum_{j=1}^{s}\alpha_j f(p_j,q_j) + \sum_{k=1}^{m}\sum_{j=1}^{s}\beta_j^{(k)}\sigma^k(p_j,q_j)J_k, \\ Q_{n+1} &= Q_n + h\sum_{j=1}^{s}\alpha_j g(p_j,q_j) + \sum_{k=1}^{m}\sum_{j=1}^{s}\beta_j^{(k)}\gamma^k(p_j,q_j)J_k \end{aligned} \tag{2.14}$$

for $n = 1,\cdots,N-1$ with $P_0 = p$, $Q_0 = q$ and

$$\begin{aligned} p_i &= P_n + h\sum_{j=1}^{s}A_{ij} f(p_j,q_j) + \sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^l\sigma^l(p_j,q_j)J_l, \\ q_i &= Q_n + h\sum_{j=1}^{s}A_{ij} g(p_j,q_j) + \sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^l\gamma^l(p_j,q_j)J_l, \end{aligned} \tag{2.15}$$

where $i = 1,\cdots,s$. These SRK methods can be characterized by the tableau

$$\begin{array}{|c|c|c|c|c} A & B^1 & B^2 & \cdots & B^k \\ \hline \alpha^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T & \cdots & {\beta^{(k)}}^T \end{array}$$

Correspondingly, we present the following theorem without proof, since the proof is similar to that of Theorem 2.1.

Theorem 2.2. *For SHS* (1.2) *and* (2.1), *if the coefficients of SRK methods* (2.14) *and* (2.15) *satisfy*

$$\alpha_i\alpha_j - \alpha_j A_{ji} - \alpha_i A_{ij} = 0, \tag{2.16}$$

$$\alpha_i\beta_j^{(k)} - \beta_j^{(k)} A_{ji} - \alpha_i B_{ij}^k = 0, \tag{2.17}$$

$$\beta_i^{(k)}\beta_j^{(l)} - \beta_j^{(l)} B_{ji}^k - \beta_i^{(k)} B_{ij}^l = 0 \tag{2.18}$$

for all $i,j = 1,\cdots,s$, $l,k = 1,\cdots,m$, *then it preserve symplectic structure, i.e.,* $dP_{n+1}\wedge dQ_{n+1} = dP_n\wedge dQ_n$.

Remark 2.3. For the single noise case, namely $m=1$, Ma, Ding and Ding presented the symplectic conditions of SRK methods (2.14) and (2.15) in [23]. Here we present Theorem 2.2 as a more general result.

It has been shown in [2,6] that SRK methods (2.14) and (2.15) will have strong global order 0.5 if

$$\alpha^T e=1, \quad {\beta^k}^T e=1, \quad {\beta^k}^T B^k e=\frac{1}{2} \tag{2.19}$$

for $k=1,\cdots,m$ are satisfied. In particular, for the commutative noise case, SRK methods (2.14) and (2.15) will have strong global order 1.0 if

$$\alpha^T e=1, \quad {\beta^i}^T e=1, \quad {\beta^i}^T B^j e+{\beta^j}^T B^i e=1 \tag{2.20}$$

for $i,j=1,\cdots,m$ are satisfied. Using the order conditions (2.19), (2.20) and symplectic conditions (2.16)-(2.18), we construct stochastic symplectic Runge-Kutta methods, for instance, a one-stage symplectic SRK methods with a tableau

$$\begin{array}{|c|c|c|c|c}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\hline
1 & 1 & 1 & \cdots & 1
\end{array}$$

which is said midpoint method, see [25,28]. Again, we can obtain a class of two-stage symplectic SRK methods with a tableau

$$\begin{array}{|cc|cc|cc|c|cc}
a & 0 & a & 0 & a & 0 & \cdots & a & 0 \\
2a & \frac{1}{2}-2a & 2a & \frac{1}{2}-2a & 2a & \frac{1}{2}-2a & \cdots & 2a & \frac{1}{2}-2a \\
\hline
2a & 1-2a & 2a & 1-2a & 2a & 1-2a & \cdots & 2a & 1-2a
\end{array}$$

which is said SRKs1 methods, where $a\in\mathbb{R}$. It is obvious that SRKs1 methods have strong order 0.5. In particular, they have strong order 1.0 for the commutative noise case.

For strong order SRK methods, in order to avoid unboundedness of absolute moments of the numerical solution, following [25], we introduce the truncated random variable for numerical solutions of stochastic Hamiltonian systems, defined by

$$\hat{J}(t,h)=\begin{cases} \tilde{J}(t,h), & |\tilde{J}(t,h)|\leq A_h, \\ A_h, & \tilde{J}(t,h)>A_h, \\ -A_h, & \tilde{J}(t,h)<-A_h, \end{cases} \tag{2.21}$$

where $\tilde{J}(t,h)=\Delta w(t)/\sqrt{h}, \Delta w(t)=w(t+h)-w(t)$. For the given $A_h=\sqrt{2k|\ln h|}$ $(k\geq 1)$, it holds that

$$E[(\hat{J}(t,h)-\tilde{J}(t,h))^2]\leq h^k, \qquad k\geq 1.$$

We refer to [5] for further information on the application of the truncated random variable to solving stochastic differential equations.

3 The symplecticity of SRK methods for SHS with special Hamiltonians

In this section we will consider a special case of the Hamiltonian system (1.2), (2.1) such that

$$H_0(t,p,q) = V_0(p) + U_0(t,q), \qquad H_k(t,p,q) = U_k(t,q), \quad k = 1,\cdots,m.$$

In this case we get the following system in the sense of Stratonovich:

$$\begin{aligned} &dP = f(t,Q)dt + \sum_{k=1}^{m} \sigma_k(t,Q) \circ dw_t^k, \qquad P(t_0) = p, \\ &dQ = g(P)dt, \qquad Q(t_0) = q \end{aligned} \tag{3.1}$$

with

$$f^i = -\frac{\partial H_0}{\partial Q^i}, \quad g^i = \frac{\partial H_0}{\partial P^i}, \quad \sigma_k^i = -\frac{\partial H_k}{\partial Q^i}, \quad i = 1,\cdots,d, \quad k = 1,\cdots,m.$$

SHS (3.1) is investigated in [25, 27, 28]. It is obvious that the system (3.1) has the same form in the sense of Itô.

For $V_0(p) = \frac{1}{2}(M^{-1}p,p)$ with M a constant, symmetric, invertible matrix, the system (3.1) takes the form

$$\begin{aligned} &dP = f(t,Q)dt + \sum_{k=1}^{m} \sigma_k(t,Q) dw_t^k, \qquad P(t_0) = p, \\ &dQ = M^{-1}Pdt, \qquad Q(t_0) = q. \end{aligned}$$

This system can be written as a second-order differential equation with multiplicative noise. Some physical applications of stochastic symplectic integration for such systems are discussed in [38].

3.1 Symplectic conditions of weak order SRK methods

For SDEs (1.1) in the sense of Itô, a class of SRK methods with $Y_0 = x_0$ is given by Rößler [34]

$$\begin{aligned} Y_{n+1} = Y_n &+ \sum_{i=1}^{s} \alpha_i a\big(t_n + c_i^{(0)}h, G_i^{(0)}\big)h \\ &+ \sum_{k=1}^{m}\sum_{i=1}^{s} \beta_i^{(1)} b^k\big(t_n + c_i^{(1)}h, G_i^{(k)}\big)\hat{I}_k + \sum_{k=1}^{m}\sum_{i=1}^{s} \beta_i^{(2)} b^k\big(t_n + c_i^{(1)}h, G_i^{(k)}\big)\frac{\hat{I}_{kk}}{\sqrt{h}} \\ &+ \sum_{k=1}^{m}\sum_{i=1}^{s} \beta_i^{(3)} b^k\big(t_n + c_i^{(2)}h, \hat{G}_i^{(k)}\big)\hat{I}_k + \sum_{k=1}^{m}\sum_{i=1}^{s} \beta_i^{(4)} b^k\big(t_n + c_i^{(1)}h, \hat{G}^{(k)}\big)\sqrt{h} \end{aligned} \tag{3.2}$$

for $n=0,1,\cdots,N-1$ with supporting values

$$G_i^{(0)}=Y_n+\sum_{j=1}^{s}A_{ij}^0 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0 b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)\hat{I}_l,$$

$$G_i^{(k)}=Y_n+\sum_{j=1}^{s}A_{ij}^1 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{j=1}^{s}B_{ij}^1 b^l\big(t_n+c_j^{(1)}h,G_j^{(k)}\big)\sqrt{h},$$

$$\hat{G}_i^{(k)}=Y_n+\sum_{j=1}^{s}A_{ij}^2 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{\substack{l=1\\l\neq k}}^{m}\sum_{j=1}^{s}B_{ij}^2 b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)\frac{\hat{I}_{kl}}{\sqrt{h}}$$

for $i=1,\cdots,s$ and $k=1,\cdots,m$. The random variables are defined by

$$\hat{I}_{kl}=\begin{cases}\frac{1}{2}(\hat{I}_k\hat{I}_l-\sqrt{h}\tilde{I}_k) & \text{if} \quad k<l,\\ \frac{1}{2}(\hat{I}_k\hat{I}_l+\sqrt{h}\tilde{I}_l) & \text{if} \quad l<k,\\ \frac{1}{2}(\hat{I}_k^2-h) & \text{if} \quad k=l\end{cases} \tag{3.3}$$

for $1\leq k,l\leq m$ with independent random variables $\hat{I}_k$, $1\leq k\leq m$, satisfy (2.4) and random variables $\tilde{I}_k$, $1\leq k\leq m-1$, satisfy (2.5). Applying SRK methods (3.2) with $\beta_i^{(2)}=0$, $\beta_i^{(3)}=0$, $\beta_i^{(4)}=0$, $i=1,\cdots,s$ to SHS (3.1), we obtain

$$\begin{aligned}P_{n+1}&=P_n+h\sum_{i=1}^{s}\alpha_i f\big(t_n+c_i^{(0)}h,q_i^{(0)}\big)+\sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\sigma^k\big(t_n+c_i^{(1)}h,q_i^{(k)}\big)\hat{I}_k,\\ Q_{n+1}&=Q_n+h\sum_{i=1}^{s}\alpha_i g\big(p_i^{(0)}\big)\end{aligned} \tag{3.4}$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$\begin{aligned}p_i^{(0)}&=P_n+h\sum_{j=1}^{s}A_{ij}^0 f\big(t_n+c_j^{(0)}h,q_j^{(0)}\big)+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0\sigma^l\big(t_n+c_i^{(1)}h,q_j^{(l)}\big)\hat{I}_l,\\ q_i^{(0)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^0 g\big(p_j^{(0)}\big),\\ p_i^{(k)}&=P_n+h\sum_{j=1}^{s}A_{ij}^1 f\big(t_n+c_j^{(0)}h,q_j^{(0)}\big)+\sum_{j=1}^{s}B_{ij}^1\sigma^l\big(t_n+c_j^{(1)}h,q_j^{(k)}\big)\hat{I}_k,\\ q_i^{(k)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^1 g\big(p_j^{(0)}\big).\end{aligned} \tag{3.5}$$

These SRK methods can be characterized by the tableau

$$\begin{array}{c|c|c|}c^{(0)} & A^0 & B^0 \\ \hline c^{(1)} & A^1 & B^1 \\ \hline & \alpha^T & {\beta^{(1)}}^T\end{array}$$

Accordingly, we present the following theorem without proof.

Theorem 3.1. *For SHS* (3.1) *and* (2.1)*, if the coefficients of SRK methods* (3.4) *and* (3.5) *satisfy* (2.8) *and* (2.9) *for all* $i,j=1,\cdots,s$*, then it preserve symplectic structure, i.e.,* $dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n$.

It has been shown in [34] that SRK methods (3.4)-(3.5) will have weak global order 1.0 if

$$\alpha^T e=1, \qquad (\beta^{(1)^T} e)^2=1, \qquad \beta^{(1)^T} B^1 e=0 \tag{3.6}$$

are satisfied. Using both the order conditions (3.6) and symplectic conditions (2.8)-(2.9), we construct stochastic symplectic Runge-Kutta methods, for instance, six weak order 1.0 one-stage SRK methods with tableaus

$$\begin{array}{c|c|c|}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 1 & 1\end{array} \qquad \begin{array}{c|c|c|}\frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & 1 & 0 \\ \hline & 1 & 1\end{array} \qquad \begin{array}{c|c|c|}\frac{1}{2} & \frac{1}{2} & 1 \\ \hline 0 & 0 & 0 \\ \hline & 1 & 1\end{array}$$

$$\begin{array}{c|c|c|}\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \hline \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 1 & -1\end{array} \qquad \begin{array}{c|c|c|}\frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & 1 & 0 \\ \hline & 1 & -1\end{array} \qquad \begin{array}{c|c|c|}\frac{1}{2} & \frac{1}{2} & -1 \\ \hline 0 & 0 & 0 \\ \hline & 1 & -1\end{array}$$

which is said SRKw3a method, SRKw3b method, SRKw3c method, SRKw3d method, SRKw3e method and SRKw3f method, respectively, in turn.

Next, let us consider the single noise case, namely $m=1$ in (1.1) and (3.1). For SDEs (1.1) in the sense of Itô, a class of SRK methods is given by Rößler [30]

$$\begin{aligned} Y_{n+1}=&Y_n+\sum_{i=1}^{s}\alpha_i a\big(t_n+c_i^{(0)}h,G_i^{(0)}\big)h \\ &+\sum_{i=1}^{s}\beta_i^{(1)}b\big(t_n+c_i^{(1)}h,G_i^{(1)}\big)\hat{I}_1+\sum_{i=1}^{s}\beta_i^{(2)}b\big(t_n+c_i^{(1)}h,G_i^{(1)}\big)\frac{\hat{I}_{11}}{\sqrt{h}} \end{aligned} \tag{3.7}$$

for $n=0,1,\cdots,N-1$ with supporting values

$$\begin{aligned} G_i^{(0)}&=Y_n+\sum_{j=1}^{s}A_{ij}^0 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{j=1}^{s}B_{ij}^0 b^l\big(t_n+c_j^{(1)}h,G_j^{(1)}\big)\hat{I}_1, \\ G_i^{(1)}&=Y_n+\sum_{j=1}^{s}A_{ij}^1 a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{j=1}^{s}B_{ij}^1 b^l\big(t_n+c_j^{(1)}h,G_j^{(1)}\big)\sqrt{h} \end{aligned}$$

for $i=1,2,\cdots,s$, where $\hat{I}_k$ and $\hat{I}_{kk}$ with $k=1$ satisfy (2.4) and (3.3), respectively. Notice that here we replace I_1 and I_{11} by $\hat{I}_1$ and $\hat{I}_{11}$, respectively, without decreasing the weak order of the method. Applying SRK methods (3.7) to SHS (3.1), we obtain

$$\begin{aligned} P_{n+1}=&P_n+h\sum_{i=1}^{s}\alpha_i f(t_n+c_i^{(0)}h,q_i^{(0)})+\sum_{i=1}^{s}\beta_i^{(1)}\sigma\big(t_n+c_i^{(1)}h,q_i^{(1)}\big)\hat{I}_1\\ &+\sum_{i=1}^{s}\beta_i^{(2)}\sigma\big(t_n+c_i^{(1)}h,q_i^{(1)}\big)\frac{\hat{I}_{11}}{\sqrt{h}},\\ Q_{n+1}=&Q_n+h\sum_{i=1}^{s}\alpha_i g(p_i^{(0)}) \end{aligned}\tag{3.8}$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$\begin{aligned} p_i^{(0)}&=P_n+h\sum_{j=1}^{s}A_{ij}^0 f(t_n+c_j^{(0)}h,q_j^{(0)})+\sum_{j=1}^{s}B_{ij}^0\sigma\big(t_n+c_i^{(1)}h,q_j^{(1)}\big)\hat{I}_1,\\ q_i^{(0)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^0 g(p_j^{(0)}),\\ p_i^{(1)}&=P_n+h\sum_{j=1}^{s}A_{ij}^1 f(t_n+c_j^{(0)}h,q_j^{(0)})+\sum_{j=1}^{s}B_{ij}^1\sigma\big(t_n+c_j^{(1)}h,q_j^{(1)}\big)\sqrt{h},\\ q_i^{(1)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^1 g(p_j^{(0)}). \end{aligned}\tag{3.9}$$

These SRK methods can be characterized by the tableau

$$\begin{array}{c|c|c|c} c^{(0)} & A^0 & B^0 & \\ \hline c^{(1)} & A^1 & B^1 & \\ \hline & \alpha^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T \end{array}$$

Accordingly, we present the following theorem without proof.

Theorem 3.2. *For SHS* (3.1) *and* (2.1) *with a single noise, if the coefficients of SRK methods* (3.8) *and* (3.9) *satisfy*

$$\alpha_i\alpha_j-\alpha_jA_{ji}^0-\alpha_iA_{ij}^0=0,\tag{3.10}$$

$$\alpha_i\beta_j^{(1)}-\alpha_iB_{ij}^0-\beta_j^{(1)}A_{ji}^1=0,\tag{3.11}$$

$$\alpha_i\beta_j^{(2)}-\beta_j^{(2)}A_{ji}^1=0\tag{3.12}$$

for all $i,j=1,\cdots,s$*, then it preserve symplectic structure, i.e.,* $dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n$.

It has been shown in [30] that SRK methods (3.8)-(3.9) will have weak global order 2.0 if

$$
\begin{aligned}
&1.\ \ \alpha^T e=1, \qquad 2.\ \ {\beta^{(2)}}^T e=0, \qquad 3.\ \ {\beta^{(1)}}^T B^1 e=0, \qquad 4.\ \ ({\beta^{(1)}}^T e)^2=1,\\
&5.\ \ {\beta^{(2)}}^T (B^1(B^1 e))=0, \qquad 6.\ \ \alpha^T A^0 e=\frac{1}{2}, \qquad 7.\ \ {\beta^{(1)}}^T (B^1(B^1(B^1 e)))=0,\\
&8.\ \ {\beta^{(1)}}^T ((B^1 e)(A^1 e))=0, \qquad 9.\ \ {\beta^{(1)}}^T (B^1(B^1 e)^2)=0, \qquad 10.\ \ {\beta^{(1)}}^T (B^1(B^1 e))=0,\\
&11.\ \ ({\beta^{(1)}}^T e)({\beta^{(1)}}^T (B^1 e)^2)=\frac{1}{2}, \qquad 12.\ \ {\beta^{(1)}}^T (B^1(A^1(B^0 e)))=0,\\
&13.\ \ \alpha^T ((B^0 e)(B^0(B^1 e)))=0, \qquad 14.\ \ {\beta^{(1)}}^T ((B^1 e)(A^0(B^1 e)))=0,\\
&15.\ \ {\beta^{(1)}}^T (A^1(B^0(B^1 e)))=0, \qquad 16.\ \ {\beta^{(1)}}^T ((B^1 e)(B^1(B^1 e)))=0,\\
&17.\ \ {\beta^{(1)}}^T (B^1 e)^3=0, \qquad 18.\ \ {\beta^{(1)}}^T (A^1(B^0 e))=0, \qquad 19.\ \ \alpha^T (B^0(B^1 e))=0,\\
&20.\ \ {\beta^{(1)}}^T (B^1(A^1 e))=0, \qquad 21.\ \ ({\beta^{(1)}}^T e)(\alpha B^0 e)=\frac{1}{2}, \qquad 22.\ \ {\beta^{(2)}}^T (A^1 e)=0,\\
&23.\ \ \alpha^T (B^0 e)^2=\frac{1}{2}, \qquad 24.\ \ {\beta^{(2)}}^T (B^1 e)^2=0, \qquad 25.\ \ {\beta^{(2)}}^T (A^1(B^0 e)^2)=0,\\
&26.\ \ {\beta^{(2)}}^T (B^1 e)=1, \qquad 27.\ \ ({\beta^{(1)}}^T e)({\beta^{(1)}}^T A^1 e)=\frac{1}{2}, \qquad 28.\ \ {\beta^{(2)}}^T (A^1(B^0 e))=0
\end{aligned}
$$

are satisfied. Using both the order conditions 1–28 and symplectic conditions (3.10)-(3.12), we construct stochastic symplectic Runge-Kutta methods, for instance, a class of weak order 2.0 three-stage SRK methods with tableau

$$
\begin{array}{c|ccc|ccc|ccc}
\frac{3}{4}-a_{21} & \frac{1}{4} & \frac{1}{2}-a_{21} & 0 & 1 & 0 & 0 & & & \\
\frac{1}{4}+a_{21} & a_{21} & \frac{1}{4} & 0 & 0 & 0 & 0 & & & \\
a_{31}+a_{32} & a_{31} & a_{32} & 0 & 1 & 1 & b_{33} & & & \\
\hline
\frac{3}{2} & 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2}+2b & -1-2b & & & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2}+b & -\frac{3}{2}-b & & & \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & b & -1-b & & & \\
\hline
 & \frac{1}{2} & \frac{1}{2} & 0 & -1 & 1 & 1 & 0 & 1 & -1
\end{array}
$$

which is said SRKw4 methods, where $a_{21},a_{31},a_{32},b_{33},b\in\mathbb{R}$.

3.2 Symplectic conditions of strong order SRK methods

For SDEs (1.1) with a single noise in the sense of Itô, a class of SRK methods with $Y_0=x_0$ is given by Küpper, Kværnø and Rößler [20]

$$
Y_{n+1}=Y_n+\sum_{i=1}^{s}\alpha_i a\big(t_n+c_i h,G_i\big)h+\sum_{i=1}^{s}\left(\beta_i^{(1)} I_1+\beta_i^{(2)}\frac{I_{11}}{\sqrt{h}}+\beta_i^{(3)}\sqrt{h}\right)b\big(t_n+c_i h,G_i\big)
$$

for $n=0,1,\cdots,N-1$ with supporting values

$$
\begin{aligned}
G_i = Y_n + &\sum_{j=1}^{s} A_{ij} a\big(t_n + c_i h, G_j\big) h \\
&+ \sum_{j=1}^{s} \left(B_{ij}^1 I_1 + B_{ij}^2 \frac{I_{11}}{\sqrt{h}} + B_{ij}^3 \sqrt{h} \right) b\big(t_n + c_i h, G_j\big)
\end{aligned}
$$

for $i=1,\cdots,s$, where $I_1=\int_{t_n}^{t_{n+1}} dw_t, I_{11}=\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dw_\tau dw_t$. Applying SRK methods (3.13) to SHS (3.1), we obtain

$$
\begin{aligned}
P_{n+1} = P_n + h&\sum_{i=1}^{s} \alpha_i f(t_n + c_i h, q_i^{(0)}) \\
&+ \sum_{i=1}^{s} \left(\beta_i^{(1)} I_1 + \beta_i^{(2)} \frac{I_{11}}{\sqrt{h}} + \beta_i^{(3)} \sqrt{h} \right) \sigma\big(t_n + c_i h, q_i^{(0)}\big), \\
Q_{n+1} = Q_n + h&\sum_{i=1}^{s} \alpha_i g(p_i^{(0)})
\end{aligned}
\tag{3.13}
$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$
\begin{aligned}
&p_i^{(0)} = P_n + h\sum_{j=1}^{s} A_{ij} f(t_n + c_j h, q_j^{(0)}) + \sum_{j=1}^{s} \left(B_{ij}^1 I_1 + B_{ij}^2 \frac{I_{11}}{\sqrt{h}} + B_{ij}^3 \sqrt{h} \right) \sigma\big(t_n + c_i h, q_j^{(0)}\big), \\
&q_i^{(0)} = Q_n + h\sum_{j=1}^{s} A_{ij} g(p_j^{(0)}).
\end{aligned}
\tag{3.14}
$$

These SRK methods can be characterized by the tableau

$$
\begin{array}{c|c|c|c|c}
c & A & B^1 & B^2 & B^3 \\
\hline
 & \alpha^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T & {\beta^{(3)}}^T
\end{array}
$$

Accordingly, we present the following theorem without proof.

Theorem 3.3. *For SHS* (3.1) *and* (2.1) *with a single noise, if the coefficients of SRK methods* (3.8) *and* (3.9) *satisfy* (2.16) *and* (2.17) *for all* $i,j=1,\cdots,s$, $k=1,2,3$, *then it preserve symplectic structure, i.e.,* $dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n$.

It has been shown in [20] that SRK methods (3.13)-(3.14) will have strong global order 1.0 if

$$
\begin{aligned}
&1.\quad \alpha^T e=1, \qquad 2.\quad {\beta^{(1)}}^T e=1, \qquad 3.\quad {\beta^{(3)}}^T e=0, \qquad 4.\quad {\beta^{(2)}}^T e=0,\\
&5.\quad {\beta^{(1)}}^T B^1 e=\frac{\lambda}{2}, \qquad 6.\quad {\beta^{(3)}}^T B^3 e=-\frac{\lambda}{2}, \qquad 7.\quad \alpha^T B^3 e=0,\\
&8.\quad {\beta^{(2)}}^T B^3 e+{\beta^{(3)}}^T B^2 e=1-\lambda, \qquad 9.\quad {\beta^{(1)}}^T B^3 e+{\beta^{(3)}}^T B^1 e=0,\\
&10.\quad {\beta^{(2)}}^T B^2 e=0, \qquad 11.\quad {\beta^{(1)}}^T B^2 e+{\beta^{(2)}}^T B^1 e=0, \qquad 12.\quad {\beta^{(3)}}^T A e=0,\\
&13.\quad 2{\beta^{(1)}}^T((B^1e)(B^2e))+2{\beta^{(1)}}^T((B^1e)(B^3e))+{\beta^{(2)}}^T(B^1e)^2+{\beta^{(2)}}^T(B^2e)^2\\
&\qquad +{\beta^{(2)}}^T((B^2e)(B^3e))+{\beta^{(3)}}^T(B^1e)^2+\frac{1}{2}{\beta^{(3)}}^T(B^2e)^2+{\beta^{(3)}}^T(B^3e)^2=0,\\
&14.\quad {\beta^{(1)}}^T(B^1(B^2e))+{\beta^{(1)}}^T(B^2(B^1e))+{\beta^{(1)}}^T(B^1(B^3e))+{\beta^{(1)}}^T(B^3(B^1e))\\
&\qquad +{\beta^{(2)}}^T(B^1(B^1e))+{\beta^{(2)}}^T(B^2(B^2e))+\frac{1}{2}{\beta^{(2)}}^T(B^2(B^3e))+\frac{1}{2}{\beta^{(2)}}^T(B^3(B^2e))\\
&\qquad +\frac{1}{2}{\beta^{(3)}}^T(B^2(B^2e))+{\beta^{(3)}}^T(B^1(B^1e))+{\beta^{(3)}}^T(B^3(B^3e))=0.
\end{aligned}
$$

Using both the order conditions 1–14 and symplectic conditions (2.16)-(2.17), we construct stochastic symplectic Runge-Kutta methods, for instance, a class of strong order 1.0 three-stage SRK methods with tableau

$$
\begin{array}{c|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & a & b & 0 & 0 & 0 & d_2-d_3 & 0 & d_3-d_2-\frac{1}{4d_1}\\
0 & 0 & 0 & 0 & a & 0 & b & 0 & 0 & 0 & d_2 & d_3 & \frac{1}{4d_1}-d_2-d_3\\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & d_1 & -d_1 & 0\\
\hline
 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & d_1 & -d_1 & 0
\end{array}
$$

which is said SRKs2 methods, where $a,b,d_1,d_2,d_3\in\mathbb{R}$, $d_1\neq 0$. Comparing with symplectic Runge-Kutta methods in [23], because of the larger amount of calculation, in theory, SRKs2 methods with strong order one are devoid of advantage in the calculations. In order to obtain more effective Runge-Kutta methods, we will consider another SRK methods in the following.

For SDEs (1.1) with multiple multiplicative noise in the sense of Itô, a class of SRK methods with $Y_0=x_0$ is given by Rößler [33]

$$
Y_{n+1}=Y_n+\sum_{i=1}^{s}\alpha_i a\big(t_n+c_i^{(0)}h,G_i^{(0)}\big)h+\sum_{k=1}^{m}\sum_{i=1}^{s}\big(\beta_i^{(1)}I_k+\beta_i^{(2)}\sqrt{h}\big)b^k\big(t_n+c_i^{(1)}h,G_i^{(k)}\big)
$$

for $n=0,1,\cdots,N-1$ with supporting values

$$
G_i^{(0)}=Y_n+\sum_{j=1}^{s}A_{ij}^{0}a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^{0}b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)I_l,
$$

$$
G_i^{(k)}=Y_n+\sum_{j=1}^{s}A_{ij}^{1}a\big(t_n+c_j^{(0)}h,G_j^{(0)}\big)h+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^{1}b^l\big(t_n+c_j^{(1)}h,G_j^{(l)}\big)\frac{I_{lk}}{\sqrt{h}}
$$

for $i=1,\cdots,s$ and $k=1,\cdots,m$, where $I_k=\int_{t_n}^{t_{n+1}}dw_t^k, I_{lk}=\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dw_\tau^l dw_t^k$. Applying SRK methods (3.15) to SHS (3.1), we obtain

$$
\begin{aligned}
P_{n+1}&=P_n+h\sum_{i=1}^{s}\alpha_i f(t_n+c_ih,q_i^{(0)})+\sum_{k=1}^{m}\sum_{i=1}^{s}\big(\beta_i^{(1)}I_k+\beta_i^{(2)}\sqrt{h}\big)\sigma^k\big(t_n+c_ih,q_i^{(k)}\big),\\
Q_{n+1}&=Q_n+h\sum_{i=1}^{s}\alpha_i g(p_i^{(0)})
\end{aligned}
\tag{3.15}
$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$
\begin{aligned}
p_i^{(0)}&=P_n+h\sum_{j=1}^{s}A_{ij}^0 f(t_n+c_jh,q_j^{(0)})+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0\sigma^l\big(t_n+c_j^{(1)}h,q_j^{(l)}\big)I_l,\\
q_i^{(0)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^0 g(p_j^{(0)}),\\
p_i^{(k)}&=P_n+h\sum_{j=1}^{s}A_{ij}^1 f(t_n+c_jh,q_j^{(0)})+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^1\sigma^l\big(t_n+c_j^{(1)}h,q_j^{(l)}\big)\frac{I_{lk}}{\sqrt{h}},\\
q_i^{(k)}&=Q_n+h\sum_{j=1}^{s}A_{ij}^1 g(p_j^{(0)}).
\end{aligned}
\tag{3.16}
$$

These SRK methods can be characterized by the tableau

$$
\begin{array}{c|c|c|c}
c^{(0)} & A^0 & B^0 & \\
\hline
c^{(1)} & A^1 & B^1 & \\
\hline
 & \alpha^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T
\end{array}
$$

Accordingly, we present the following theorem without proof.

Theorem 3.4. *For SHS* (3.1) *and* (2.1)*, if the coefficients of SRK methods* (3.15) *and* (3.16) *satisfy* (3.10)*,* (3.11) *and* (3.12) *for all $i,j=1,\cdots,s$, then it preserve symplectic structure, i.e., $dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n$.*

It has been shown in [33] that SRK methods (3.15)-(3.16) will have strong global order 1.0 if

$$
\begin{array}{lll}
1.\ \ \alpha^Te=1, & 2.\ \ {\beta^{(1)}}^Te=1, & 3.\ \ {\beta^{(2)}}^TB^1e=0,\\
4.\ \ {\beta^{(1)}}^TB^1e=0, & 5.\ \ {\beta^{(2)}}^TB^1e=1, & 6.\ \ {\beta^{(2)}}^TA^1e=0,\\
7.\ \ {\beta^{(2)}}^T(B^1e)^2=0, & 8.\ \ {\beta^{(2)}}^T(B^1(B^1e))=0. &
\end{array}
$$

Using both the order conditions 1–8 and symplectic conditions (3.10)-(3.12), we construct stochastic symplectic Runge-Kutta methods, for instance, a class of strong order 1.0 two-stage SRK methods with tableau

$$
\begin{array}{c|cc|cc|cc}
a & a & 0 & 0 & 0 & & \\
\frac{1}{2}+a & 2a & \frac{1}{2}-a & 0 & 0 & & \\
\hline
1 & 2a & 1-2a & b_1 & b_2 & & \\
1 & 2a & 1-2a & -b_2 & b_1 & & \\
\hline
 & 2a & 1-2a & \frac{b_2-b_1}{2b_2} & \frac{b_2+b_1}{2b_2} & \frac{1}{2(b_1+b_2)} & -\frac{1}{2(b_1+b_2)}
\end{array}
$$

which is said SRKs3 methods, where $a \in \mathbb{R}, b_2 \neq 0, b_1+b_2 \neq 0$. It shown that we have found a class of strong 1.0 order symplectic Runge-Kutta methods for SHS with multiple multiplicative noise.

Remark 3.1. SRK methods (3.15) also convergent to the solution of Stratonovich SDEs (1.1), see [33], but under the corresponding order conditions these SRK methods can not preserve symplectic structure for SHS (1.2) with general nonseparable Hamiltonians.

4 The symplecticity of SRK methods for SHS with multiple additive noise

In this section we will consider the SHS with multiple additive noise, namely

$$
\begin{aligned}
dP &= f(t,P,Q)dt + \sum_{k=1}^{m} \sigma_k(t) dw_t^k, \qquad P(t_0)=p, \\
dQ &= g(t,P,Q)dt + \sum_{k=1}^{m} \gamma_k(t) dw_t^k, \qquad Q(t_0)=q
\end{aligned} \tag{4.1}
$$

with

$$
f^i = -\frac{\partial H}{\partial Q^i}, \quad g^i = \frac{\partial H}{\partial P^i} \quad i=1,\cdots,d,
$$

where P,Q,p,q are d-dimensional vectors. Such SHS can be written as the form as following SDEs

$$
dX_t = a(t,X_t)dt + \sum_{k=1}^{m} b_k(t) dw_t^k, \qquad X_{t_0} = x_0, \tag{4.2}
$$

where $X, a(t,X)$ are r-dimensional column-vectors, and $r=2d$.

4.1 Symplectic conditions of high strong order SRK methods

For SDEs (4.2) with multiple noise in the sense of Itô, a class of SRK methods with $Y_0 = x_0$ is given by Rößler [33]

$$
\begin{aligned}
Y_{n+1} = Y_n &+ h \sum_{i=1}^{s} \alpha_i a(t_n + c_i^{(0)} h, G_i) \\
&+ \sum_{k=1}^{m} \sum_{i=1}^{s} \left(\beta_i^{(1)} I_k + \beta_i^{(2)} \frac{I_{k0}}{h} \right) b^k(t_n + c_i^{(1)} h)
\end{aligned} \tag{4.3}
$$

for $n=0,1,\cdots,N-1$ with stage values

$$G_i=Y_n+h\sum_{j=1}^{s}A_{ij}a(t_n+c_j^{(0)}h,G_j)+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}b^l(t_n+c_j^{(1)}h)\frac{I_{l0}}{h}$$

for $i=1,\cdots,s$, where $I_{l0}=\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}dw_{s_1}^l ds$. Applying SRK methods (4.3) to SHS (4.1), we obtain

$$\begin{aligned}P_{n+1}&=P_n+h\sum_{i=1}^{s}\alpha_i f(t_n+c_i^{(0)}h,p_i^{(0)},q_i^{(0)})+\sum_{k=1}^{m}\sum_{i=1}^{s}\big(\beta_i^{(1)}I_k+\beta_i^{(2)}\frac{I_{k0}}{h}\big)\sigma^k\big(t_n+c_i^{(1)}h\big),\\ Q_{n+1}&=Q_n+h\sum_{i=1}^{s}\alpha_i g(t_n+c_i^{(0)}h,p_i^{(0)},q_i^{(0)})+\sum_{k=1}^{m}\sum_{i=1}^{s}\big(\beta_i^{(1)}I_k+\beta_i^{(2)}\frac{I_{k0}}{h}\big)\gamma^k\big(t_n+c_i^{(1)}h\big)\end{aligned} \tag{4.4}$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$\begin{aligned}p_i^{(0)}&=P_n+h\sum_{j=1}^{s}A_{ij}f(t_n+c_j^{(0)}h,p_j^{(0)},q_j^{(0)})+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}\sigma^l\big(t_n+c_j^{(1)}h\big)\frac{I_{l0}}{h},\\ q_i^{(0)}&=Q_n+h\sum_{j=1}^{s}A_{ij}g(t_n+c_j^{(0)}h,p_j^{(0)},q_j^{(0)})+\sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}\gamma^l\big(t_n+c_j^{(1)}h\big)\frac{I_{l0}}{h}\end{aligned} \tag{4.5}$$

for $i=1,\cdots,s$. These SRK methods can be characterized by the tableau

$$\begin{array}{c|c|c|c} c^{(0)} & A & B & c^{(1)} \\ \hline & \alpha^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T \end{array}$$

Accordingly, we present the following theorem without proof.

Theorem 4.1. *For SHS* (4.1) *and* (2.1)*, if the coefficients of SRK methods* (4.4) *and* (4.5) *satisfy*

$$\alpha_i\alpha_j-\alpha_jA_{ji}-\alpha_iA_{ij}=0 \tag{4.6}$$

for all $i,j=1,\cdots,s$, then it preserve symplectic structure, i.e., $dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n$.

It has been shown in [33] that SRK methods (4.4)-(4.5) will have strong global order 1.5 if

$$\begin{array}{lll} 1.\ \ \alpha^Te=1, & 2.\ \ {\beta^{(1)}}^Te=1, & 3.\ \ {\beta^{(2)}}^Te=0,\\ 4.\ \ \alpha^TBe=1, & 5.\ \ \alpha^TAe=\dfrac{1}{2}, & 6.\ \ \alpha^T(Be)^2=\dfrac{3}{2},\\ 7.\ \ {\beta^{(1)}}^Tc^{(1)}=1, & 8.\ \ {\beta^{(2)}}^Tc^{(1)}=-1. & \end{array}$$

Using both the order conditions 1–8 and symplectic conditions (4.6), we construct stochastic symplectic Runge-Kutta methods, for instance, a class of strong order 1.5 two-stage SRK methods with tableau

$$\begin{array}{c|cc|cc|cc}
\frac{1}{3} & \frac{1}{3} & 0 & \frac{3}{2} & 0 & 1+\frac{\beta_1-1}{\beta_2} & \\
\frac{5}{6} & \frac{2}{3} & \frac{1}{6} & b & -b & 1+\frac{\beta_1}{\beta_2} & \\
\hline
 & \frac{2}{3} & \frac{1}{3} & \beta_1 & 1-\beta_1 & \beta_2 & -\beta_2
\end{array}$$

which is said SRKs4 methods, where $\beta_1, b\in\mathbb{R}, \beta_2\neq 0$.

Remark 4.1. The software package Maple was used in the generation process of SRKw2, SRKw4, SRKs2, SRKs3 and SRKs4 methods to aid in algebraic computation. This involves the solutions of large numbers of order and symplectic conditions.

4.2 Symplectic conditions of weak order SRK methods

For SDEs (4.2) with multiple noise in the diffusion autonomous case, a class of SRK methods with $Y_0=x_0$ is given by Debrabant [11]

$$Y_{n+1} \quad = \quad Y_n+h\sum_{i=1}^{s}\alpha_i a(t_n+c_ih,G_i)+\sum_{k=1}^{m}b^k\hat{I}_k \tag{4.7}$$

for $n=0,1,\cdots,N-1$ with stage values

$$G_i=Y_n+h\sum_{j=1}^{s}A_{ij}a(t_n+c_jh,G_j)+\sum_{k=1}^{m}b^k(d_{1,i}\hat{I}_k+d_{2,i}\hat{I}_{k+m})$$

for $i=1,\cdots,s$, where b^k are constants for $k=1,\cdots,m$, $\hat{I}_k$ and $\hat{I}_{k+m}$ satisfy (2.4). Applying SRK methods (4.7) to SHS (4.1) in the diffusion autonomous case, we obtain

$$\begin{aligned}
P_{n+1}&=P_n+h\sum_{i=1}^{s}\alpha_i f(t_n+c_ih,p_i^{(0)},q_i^{(0)})+\sum_{k=1}^{m}\sigma^k\hat{I}_k,\\
Q_{n+1}&=Q_n+h\sum_{i=1}^{s}\alpha_i g(t_n+c_ih,p_i^{(0)},q_i^{(0)})+\sum_{k=1}^{m}\gamma^k\hat{I}_k
\end{aligned} \tag{4.8}$$

for $n=1,\cdots,N-1$ with $P_0=p$, $Q_0=q$ and

$$\begin{aligned}
p_i^{(0)}&=P_n+h\sum_{j=1}^{s}A_{ij}f(t_n+c_jh,p_j^{(0)},q_j^{(0)})+\sum_{k=1}^{m}\sigma^k(d_{1,i}\hat{I}_k+d_{2,i}\hat{I}_{k+m}),\\
q_i^{(0)}&=Q_n+h\sum_{j=1}^{s}A_{ij}g(t_n+c_jh,p_j^{(0)},q_j^{(0)})+\sum_{k=1}^{m}\gamma^k(d_{1,i}\hat{I}_k+d_{2,i}\hat{I}_{k+m})
\end{aligned} \tag{4.9}$$

for $i=1,\cdots,s$, where σ^k and γ^k are constants for $k=1,\cdots,m$. These SRK methods can be characterized by the tableau

$$\begin{array}{c|c|c|c}
c & A & d_1 & d_2\\
\hline
 & \alpha^T & &
\end{array}$$

Accordingly, we present the following theorem without proof.

Theorem 4.2. *For SHS* (4.1) *and* (2.1), *if the coefficients of SRK methods* (4.8) *and* (4.9) *satisfy* (4.6) *for all* $i,j=1,\cdots,s$, *then it preserve symplectic structure, i.e.,* $dP_{n+1}\wedge dQ_{n+1}=dP_n\wedge dQ_n$.

It has been shown in [11] that SRK methods (4.8)-(4.9) will have weak global order 2.0 if

$$1.\quad \alpha^T e=1,\qquad 2.\quad \alpha^T Ae=\frac{1}{2},\qquad 3.\quad \alpha^T(d_1^2+d_2^2)=\frac{1}{2},\qquad 4.\quad \alpha^T d_1=\frac{1}{2}.$$

Using both the order conditions 1–4 and symplectic conditions (4.6), we construct stochastic symplectic Runge-Kutta methods, for instance, a weak order 2.0 one-stage SRK method with tableau

$$\begin{array}{c|c|c|c}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\hline
 & 1 & &
\end{array}$$

which is said SRKw5 method.

Remark 4.2. Debrabant presented the weak third order conditions of SRK methods (4.7) in [11]. Even now, using both the weak third order conditions and symplectic conditions (4.6), we hard to find a weak third order symplectic Runge-Kutta method with a smaller stage number. For instance, we can not find it when the stage number $s<7$.

5 Numerical experiments

In this section, we applying derived symplectic Runge-Kutta methods to several numerical examples in order to confirm our theoretical results.

5.1 Example 1: A linear stochastic oscillator with additive noise

Let us consider the example given in [39]:

$$\begin{aligned} &dx(t)=y(t)dt, && x(t_0)=x_0\in\mathbb{R},\\ &dy(t)=-x(t)dt+\sigma dw_t, && y(t_0)=y_0\in\mathbb{R}, \end{aligned}\tag{5.1}$$

where $\sigma>0$ is a constant. For SDE (5.1), its Itô and Stratonovich form are identical. It is obvious that (5.1) is a stochastic Hamiltonian system with

$$H_0(x,y)=\frac{x^2+y^2}{2},\qquad H_1(x,y)=-\sigma x,$$

such that

$$y=\frac{\partial H_0}{\partial y},\qquad x=\frac{\partial H_0}{\partial x},\qquad \sigma=-\frac{\partial H_1}{\partial x},$$

so the phase flow of (5.1) preserves the symplectic structure (2.1). Stømmen and Higham [39] have shown that (5.1) has the linear growth property of the second moment, i.e.,

$$Me(t)=E(x^2(t)+y^2(t))=x_0^2+y_0^2+\sigma^2 t.$$

For symplectic Runge-Kutta methods, we want to utilise numerical tests to check their ability of preserving the linear growth property of the second moment. The coefficients of Eq. (5.1) is chosen as $\sigma=1$, $x_0=0$, $y_0=1$ and fix stepsize $h=0.2$. The second moment $E(X_n^2+Y_n^2)$ of the numerical solution is approximated by taking sample average of K sample trajectories, i.e.,

$$Me_n=\frac{1}{K}\sum_{i=1}^{K}(X_n^2(\omega_i)+Y_n^2(\omega_i)),$$

where $K=1000,000$. To compare our symplectic integrators with nonsymplectic ones, we use the Euler-Maruyama method [19,28] and Heun method [6,19]:

$$\begin{aligned}&Y=y_n+ha(t_n,y_n)+b(t_n,y_n)\Delta w_n,\\&y_{n+1}=y_n+\frac{1}{2}h(a(t_n,y_n)+a(t_n,Y))+\frac{1}{2}(b(t_n,y_n)+b(t_n,Y))\Delta w_n.\end{aligned}$$

As shown by Fig. 1, the linear growth of second moment of the numerical solutions produced by symplectic Runge-Kutta methods. The reference line (dotted) has slope 1, along which the second moment of the solution should stretch. It can be seen that there are large errors in the growth rates of the second moments by Euler-Maruyama and Heun methods. Here some coefficients of Runge-Kutta methods are chosen as the SRKw1 method with $b=0$ (SRKw1I), SRKw2 method with $b_1=b_2=b_3=0$ (SRKw2I), SRKw4 method with $a_{21}=a_{31}=a_{32}=b_{33}=b=0$ (SRKw4I), SRKs1 method with $a=\frac{1}{4}$ (SRKs1I), SRKs2 method with $a=b=0,d=0.5$ (SRKs2I), SRKs3 method with $a=\frac{1}{4},b_1=b_2=1$ (SRKs3I) and SRKs4 method with $\beta_1=b=0,\beta_2=1$ (SRKs4I). Further, we use the average of biases

$$\hat{e}_n=\frac{1}{n}|Me(t_n)-Me_n|$$

to measure the accuracy of preserving the linear growth property of the second moment. We observe Table 1 for all symplectic Runge-Kutta methods which confirm they all are much better than nonsymplectic Euler-Maruyama and Heun methods in the sense of the average of biases. Note that we observe the small errors (<0.005) of symplectic methods are due to the Monte-Carlo errors, which could be further reduced by increasing the number of samples.

Again, we consider weak convergence rates for numerical solutions of Eq. (5.1). The SRKw5 method is applied and compared with the Euler-Maruyama method. According to [39], the system (5.1) has the unique solution

$$\begin{aligned}x(t)&=x_0\cos t+y_0\sin t+\sigma\int_0^t\sin(t-s)dw_s,\\y(t)&=-x_0\sin t+y_0\cos t+\sigma\int_0^t\cos(t-s)dw_s.\end{aligned}$$

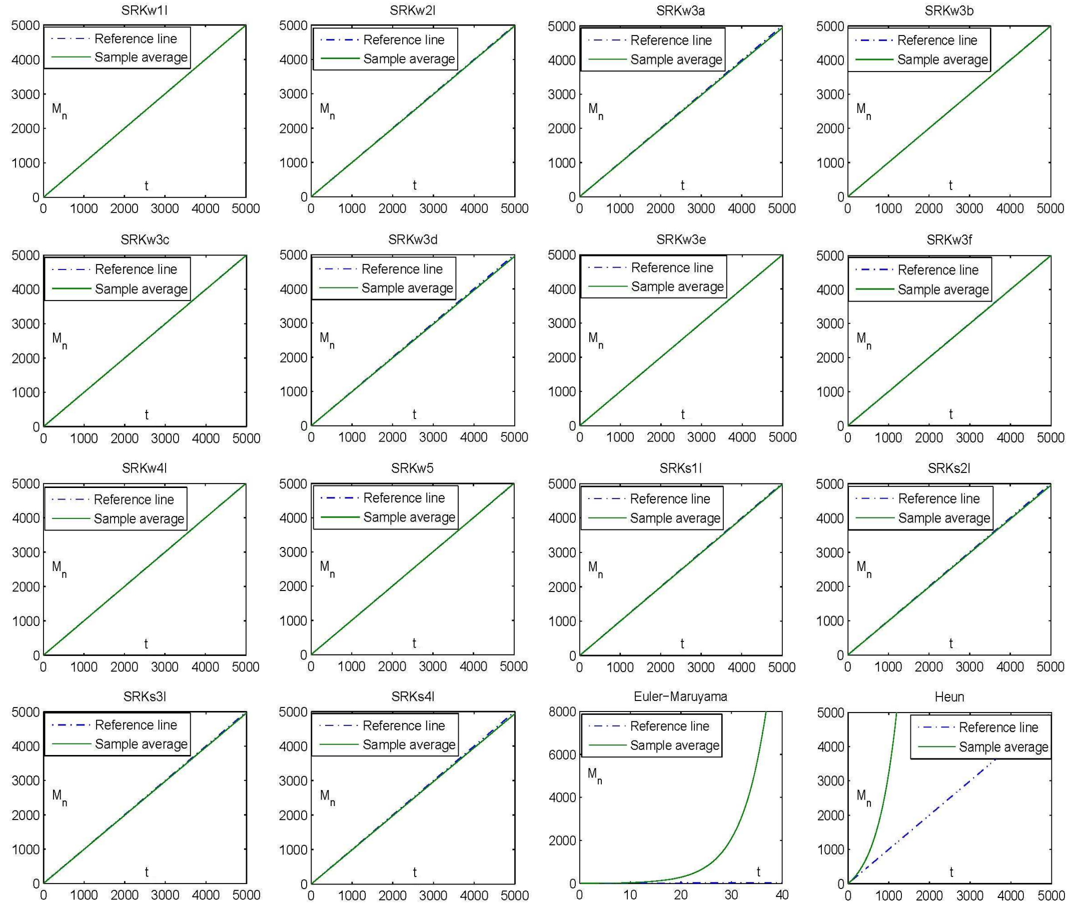


Figure 1: Second moments of approximations for Eq. (5.1).

Hence,

$$E(x^2(t)) = (x_0 \cos t + y_0 \sin t)^2 + \sigma \int_0^t \sin^2(t-s) ds.$$

In Fig. 2(a), we plot the errors for $E(x^2)$ at time $t=1$ versus the timestep $h=2^{-i}$, $i=1,\cdots,8$. To carefully check the accuracy of the methods, we numerically compute $E(x^2)$ using the averages over 30 million trajectories. Further, some reference lines (broken) with slope one and two are plotted for better comparison. We also consider strong convergence rates for numerical solutions of Eq. (5.1). We arrange the simulations into M batches of K simulations in the following way. Denoting by $y_{i,j,N}$ the numerical approximation to $y_{i,j}(t_N)$ at step point t_N in the $i \times j$-th simulation of all $K \times M$ simulations, we use means of absolute errors

$$\hat{\epsilon}(y) = \frac{1}{KM} \sum_{j=1}^{M} \sum_{i=1}^{K} \sqrt{\sum_{k=1}^{r} \left(y_{i,j,N}^k - y_{i,j}^k(t_N) \right)^2}$$

Table 1: The averages of biases of numerical methods for Eq. (5.1).

Methods	5000 steps	10000 steps	25000 steps
Euler	$7.341241738e+082$	$5.403073301e+167$	Inf
Heun	0.4442603325	2.505446702	443.9279506
SRKw1I	0.002511919556	0.001959065597	0.002088948678
SRKw2I	0.004204629157	0.004158916414	0.004082373282
SRKw3a	0.002140102824	0.001982949467	0.002059087292
SRKw3b	0.00006087204307	0.0002537318069	0.00003997775495
SRKw3c	0.0001602655490	0.0001341840902	0.0002845553792
SRKw3d	0.002078596062	0.001873222360	0.001956775823
SRKw3e	0.0002981423602	0.00004757943303	0.0002170102312
SRKw3f	0.00002076197241	0.0001974424086	0.0004457843294
SRKw4I	0.0001450897061	0.0001621590728	0.0002163903355
SRKw5	0.0003362889021	0.00002246683947	0.00005073875569
SRKs1I	0.0008899479199	0.001168208442	0.001150954402
SRKs2I	0.002090830473	0.002016024143	0.001959391097
SRKs3I	0.002626291461	0.002448442391	0.002179862105
SRKs4I	0.002417333345	0.002448624413	0.002313668062

to measure the accuracy and strong convergence property of numerical methods, where $r=2$, $M=20$ and $K=250$. The SRKs4I method is applied and compared with the Euler-Maruyama method. In Fig. 2(b), we plot the errors for $\hat{e}(x)$ at time $t=1$ versus the timestep $h=2^{-i}$, $i=1,\cdots,8$. Some reference lines (broken) with slope 1.0 and 1.5 are plotted for better comparison.

5.2 Example 2: A stochastic harmonic oscillator

The stochastic harmonic oscillator model [29] in the sense of Stratonovich is described by

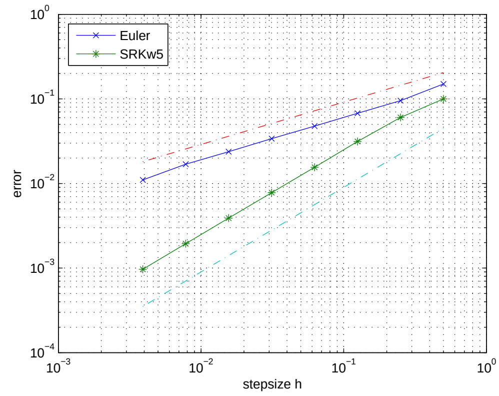
$$\begin{aligned} dx(t) &= p(t)dt+\sigma p(t)\circ dw_t, \qquad x(t_0)=x_0\in\mathbb{R}, \\ dp(t) &= -x(t)dt-\sigma x(t)\circ dw_t, \quad p(t_0)=p_0\in\mathbb{R}, \end{aligned} \tag{5.2}$$

It is obvious that (5.2) is a stochastic Hamiltonian system with

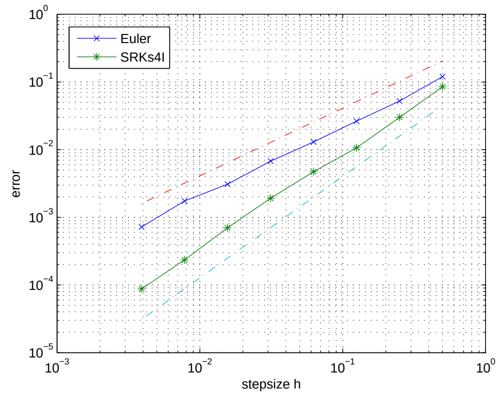
$$H_0(x,p)=\frac{1}{2}(x^2+p^2), \qquad H_1(x,y)=\frac{1}{2}\sigma(x^2+p^2),$$

such that

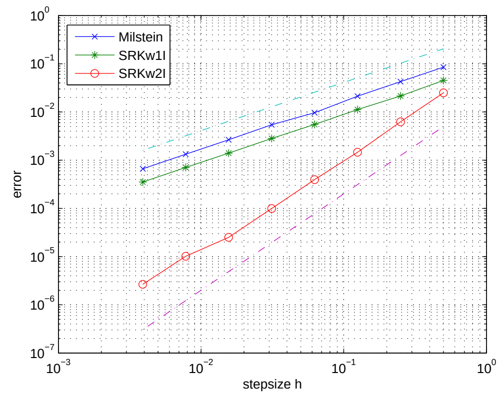
$$p=\frac{\partial H_0}{\partial p}, \qquad x=\frac{\partial H_0}{\partial x}, \qquad \sigma p=\frac{\partial H_1}{\partial p}, \qquad \sigma x=\frac{\partial H_1}{\partial x},$$



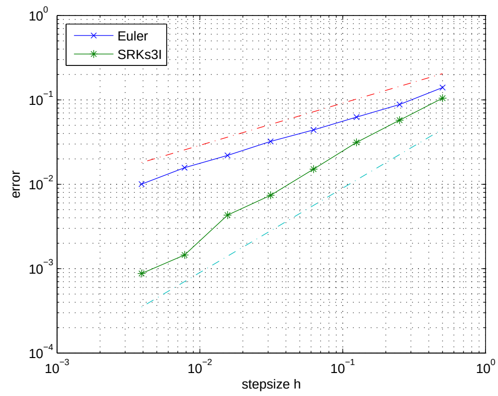
(a) Error for $E(x^2)$



(b) Means of absolute errors for x



(c) Error for $E(x^2)$



(d) Means of absolute errors for P

Figure 2: Comparison of convergence rates for numerical solutions of Eq. (5.1) (subplots (a) and (b)), Eq. (5.2) (subplot (c)) and Eq. (5.3) (subplot (d)).

so the phase flow of (5.2) preserves the symplectic structure (2.1). For this system, the energy function $H_0(x,p)$ is a conserved quantity. To compare our symplectic integrators with nonsymplectic ones, we use the Milstein method and Heun method [19, 28]. Fig. 3 gives approximations of a sample phase trajectory of (5.2) simulated by the Milstein method (top left), Heun method (top right), SRKs1I method (bottom left), SRKw1I method (bottom middle) and SRKw2I method (bottom right), respectively. The initial condition is $x_0=1$, $p_0=0$. The corresponding exact phase trajectory belongs to the circle with the center at the origin and with the unit radius. In [18], Jiang et al. presented the exact solution of (5.2)

$$\begin{aligned}
x(t) &= p_0\sin(t+\sigma w_t)+x_0\cos(t+\sigma w_t),\\
p(t) &= p_0\cos(t+\sigma w_t)-x_0\sin(t+\sigma w_t).
\end{aligned}$$

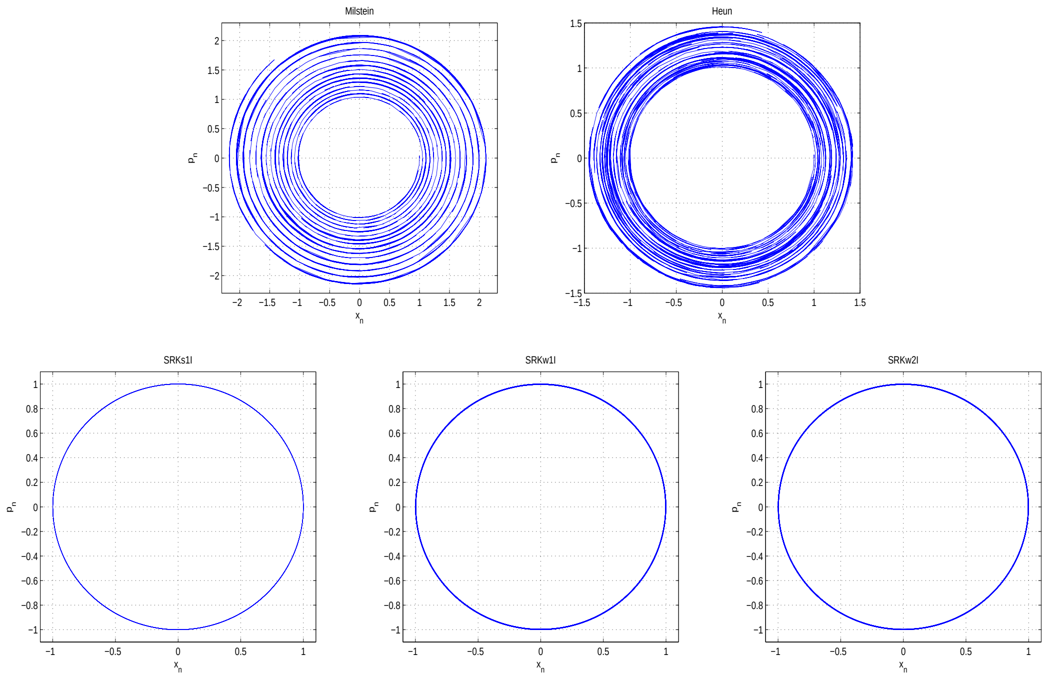


Figure 3: The numerical flows produced by the Milstein method (top left), Heun method (top right), SRKs1I method (bottom left), SRKw1I method (bottom middle) and SRKw2I method (bottom right), respectively. $T=100$, $h=0.01$, $\sigma=1$ and $(x_0,p_0)=(1,0)$ for Eq. (5.2).

Again, we consider weak convergence rates for numerical solutions of Eq. (5.2). SRKw1I and SRKw2I methods are applied and compared with the Milstein method. In Fig. 2(c), we plot the errors for $E(x^2)$ at time $t=1$ versus the timestep $h=2^{-i}$, $i=1,\cdots,8$. To carefully check the accuracy of the methods, we numerically compute $E(x^2)$ using the averages over 30 million trajectories. Further, some reference lines (broken) with slope one and two are plotted for better comparison.

5.3 Example 3: A model for synchrotron oscillations of particles in storage rings

The following model, studied in [25,27,28,38], describes synchrotron oscillations of particles in storage rings under the influence of external fluctuating electromagnetic fields,

$$\begin{aligned} &dP=-\omega^2\sin(Q)dt-\sigma_1\cos(Q)dw_t^1-\sigma_2\sin(Q)dw_t^2,\\ &dQ=Pdt. \end{aligned} \tag{5.3}$$

Approximations of a sample trajectory of (5.3) simulated by the SRKw3a, SRKw3b, SRKw3c, SRKw3d, SRKw3e, SRKw3f, SRKs3I methods and the Euler method are plotted on Fig. 4. The trajectories obtained by the numerical methods with $h=0.02$. Fig. 4

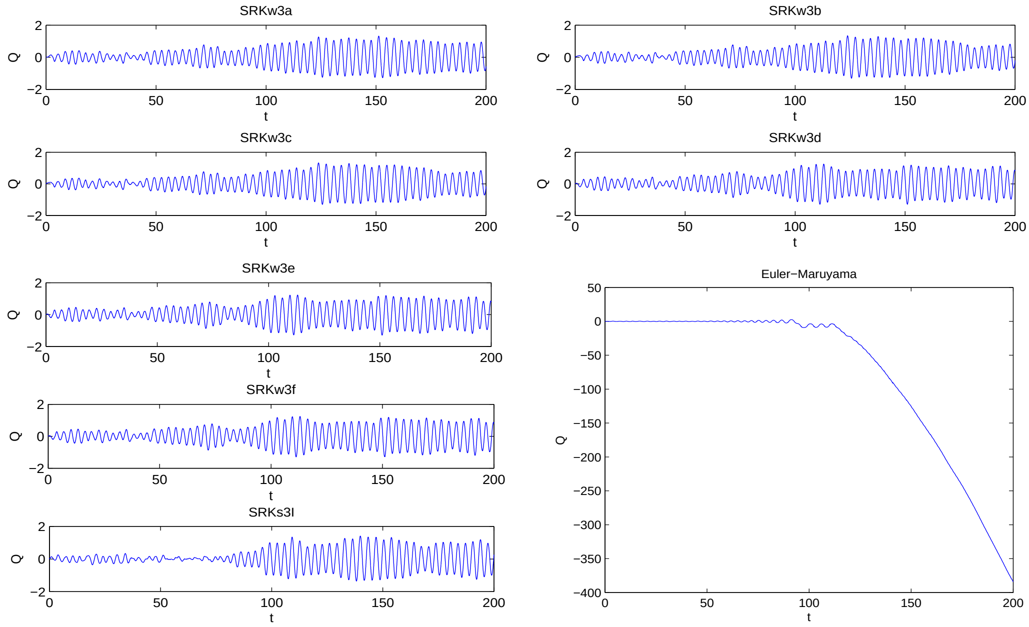


Figure 4: A sample trajectory of approximations of Eq. (5.3) for $\omega=2$, $\sigma_1=0.2$, $\sigma_2=0.1$, $h=0.02$.

shows the oscillation property of the numerical solutions produced by symplectic Runge-Kutta methods. It can be seen that the Euler method is unacceptable for simulation of the solution to (5.3) on a long time interval.

Again, we consider strong convergence rates for numerical solutions of Eq. (5.3) with $\omega=2$, $\sigma_1=0.2$ and $\sigma_2=0.1$. The SRKs3I method is applied and compared with the Euler-Maruyama method. We have carefully implemented the above integrators in Matlab. In Fig. 2(d), we plot the means of absolute errors for P at time $t=1$ versus the timestep $h=2^{-i}$, $i=1,\cdots,8$. The reference solution is computed using the small timestep $h=2^{-14}$. The error is calculated based on M batches with K trajectories in each, where $M=20$ and $K=250$. Further, some reference lines (broken) with slope one-half and one are plotted for better comparison. We also consider weak convergence rates for numerical solutions of Eq. (5.3) with $\omega=2$, $\sigma_1=0.2$ and $\sigma_2=0$. SRKw3a and SRKw4I methods are applied and compared with the Euler-Maruyama method. In Fig. 6(a), we plot the errors for $E(P^2)$ at time $t=1$ versus the timestep $h=2^{-i}$, $i=1,\cdots,8$. The reference solution is computed using the small timestep $h=2^{-14}$. To carefully check the accuracy of the methods, we numerically compute $E(P^2)$ using the averages over 20 million trajectories. Further, some reference lines (broken) with slope one and two are plotted for better comparison.

5.4 Example 4: A stochastic rigid body model

Finally, we consider a randomly perturbed rigid body problem, i.e., the motion of a rigid body in $\mathbb{R}^3$ subject to a single Stratonovich noise perturbation [1,7,21,22,24]

$$\begin{aligned} dX &= \widehat{X}\mathcal{I}Xdt + \mu\widehat{X}e_1 \circ dw_t, \\ dQ &= Q\widehat{\mathcal{I}X}dt + \mu Q\widehat{e_1} \circ dw_t, \end{aligned} \tag{5.4}$$

where

$$\widehat{X} = \begin{pmatrix} 0 & -X_{[3]} & X_{[2]} \\ X_{[3]} & 0 & -X_{[1]} \\ -X_{[2]} & X_{[1]} & 0 \end{pmatrix}$$

for all $X = (X_{[1]}, X_{[2]}, X_{[3]})^T$, $e_1 = (1,0,0)^T$, $\mu \geq 0$ is a parameter, $\mathcal{I} = \mathrm{diag}(I_1, I_2, I_3)$, and the constants $I_1, I_2, I_3 > 0$ are the moments of inertia which characterize the rigid body. In the case where $\mu = 0$, we recover the standard deterministic equations of motion of an asymmetric rigid body [13]. Notice that the function $X(t)$ represents the angular momentum in $\mathbb{R}^3$ in the body frame, and it satisfies

$$\begin{aligned} dX_{[1]} &= \Big(\frac{1}{I_3} - \frac{1}{I_2}\Big) X_{[2]} X_{[3]} dt, \\ dX_{[2]} &= \Big(\frac{1}{I_1} - \frac{1}{I_3}\Big) X_{[3]} X_{[1]} dt + \mu X_{[3]} \circ dw_t, \\ dX_{[3]} &= \Big(\frac{1}{I_2} - \frac{1}{I_1}\Big) X_{[1]} X_{[2]} dt - \mu X_{[2]} \circ dw_t. \end{aligned}$$

SRKw1I, SRKw2I, and SRKs1I methods are applied and compared with the Milstein and Heun methods. We set $T = 40$, $h = 0.02$ and $(X_{[1]}(0), X_{[2]}(0), X_{[3]}(0)) = (\cos(1.1), 0, \sin(1.1))$, and $I_1 = 0.8$, $I_2 = 0.6$, $I_3 = 0.2$ and $\mu = 0.1$. It can be observed from Fig. 5 that the numerical solutions produced by Milstein and Heun methods show an unacceptable qualitative behaviour and even drift away form the sphere, while the numerical solutions obtained by SRKw1I, SRKw2I, and SRKs1I methods lie on the sphere exactly as we expected.

Again, we consider weak convergence rates for numerical solutions of Eq. (5.4). SRKw1I and SRKw2I methods are applied and compared with the Milstein method. In Fig. 6(b), we plot the errors for $E(X_{[1]}^2)$ at time $t = 1$ versus the timestep $h = 2^{-i}$, $i = 1, \cdots, 8$. The reference solution is computed using the small timestep $h = 2^{-14}$. To carefully check the accuracy of the methods, we numerically compute $E(X_{[1]}^2)$ using the averages over 20 million trajectories. Further, some reference lines (broken) with slope one and two are plotted for better comparison. Note that for small timesteps ($h < 0.125$) the zigzag that we observe is due to the Monte-Carlo error, which could be further reduced by increasing the number of samples.

6 Conclusion

In this paper, we have investigated stochastic symplectic Runge-Kutta methods for three types of SHS. We gave conditions to construct SRK methods preserving the sym-

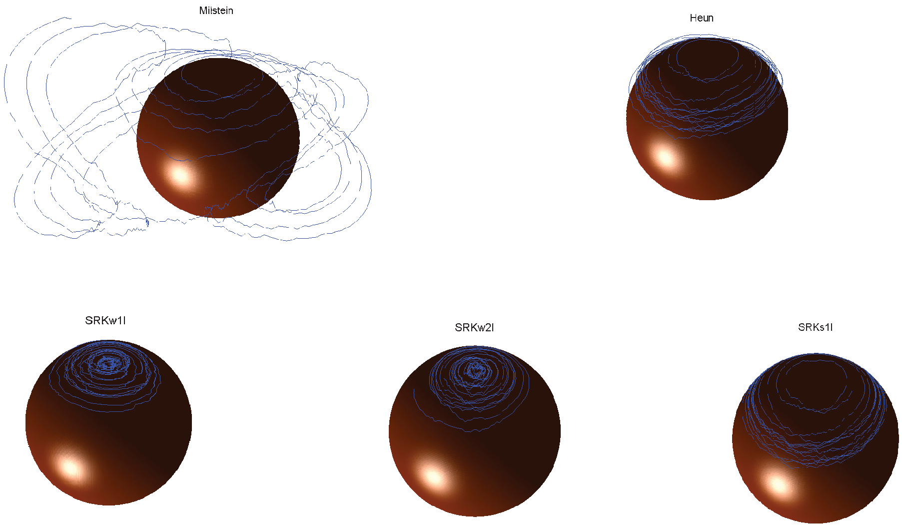
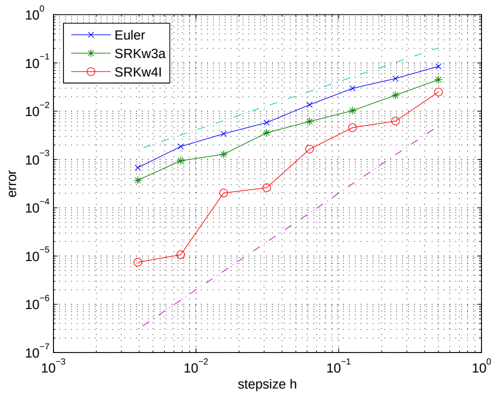
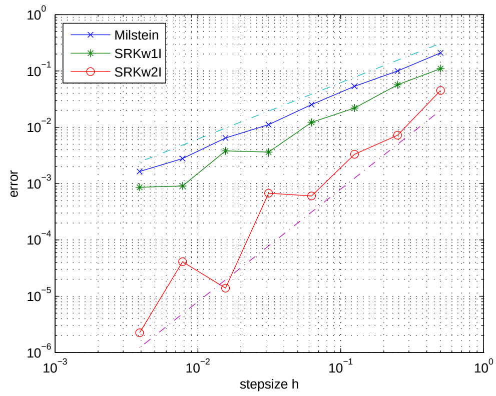


Figure 5: A sample trajectory of approximations of Eq. (5.4) for the stochastic rigid body with noise size $\mu=0.1$.



(a) Error for $E(P^2)$.



(b) Error for $E(X_{[1]}^2)$.

Figure 6: Comparison of weak convergence rates for numerical solutions of Eq. (5.3) (subplot (a)) and Eq. (5.4) (subplot (b)).

plectic property with weak and strong convergence order, respectively. Based on the weak/strong order and symplectic conditions, some effective schemes are derived. In particular, we obtained two classes of high weak order symplectic Runge-Kutta methods for SHS with a single multiplicative noise, and two classes of high strong order symplectic Runge-Kutta methods for SHS with multiple multiplicative and additive noise, respectively, with the help of computer algebra. This involves the solutions of large numbers of order and symplectic conditions. Four stochastic models are tested to verify our analysis and show that the schemes have good long time behaviour as expected in the simulation. Future work will consider constructing high order symplectic Runge-Kutta methods for nonseparable SHS with multiple multiplicative noise, for instance, strong order 1.0 methods or weak order 2.0 methods.

Appendix: Proof of Theorem 2.1

Introduce the temporary notations

$$\begin{aligned} &f_i = f(t_n + c_i^{(0)}h, p_i^{(0)}, q_i^{(0)}), \quad g_i = g(t_n + c_i^{(0)}h, p_i^{(0)}, q_i^{(0)}), \\ &\sigma_i^k = \sigma^k(t_n + c_i^{(1)}h, p_i^{(k)}, q_i^{(k)}), \quad \gamma_i^k = \gamma^k(t_n + c_i^{(1)}h, p_i^{(k)}, q_i^{(k)}). \end{aligned}$$

Differentiating (2.6), we obtain

$$\begin{aligned} dP_{n+1} &= dP_n + h\sum_{i=1}^{s}\alpha_i df_i + \sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\hat{I}_k d\sigma_i^k, \\ dQ_{n+1} &= dQ_n + h\sum_{i=1}^{s}\alpha_i dg_i + \sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\hat{I}_k d\gamma_i^k. \end{aligned}$$

From the exterior products, we have

$$\begin{aligned} dP_{n+1}\wedge dQ_{n+1} = &\, dP_n\wedge dQ_n + h\sum_{i=1}^{s}\alpha_i dP_n\wedge dg_i + \sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\hat{I}_k dP_n\wedge d\gamma_i^k \\ &+ h\sum_{i=1}^{s}\alpha_i df_i\wedge dQ_n + \sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\hat{I}_k d\sigma_i^k\wedge dQ_n \\ &+ h^2\sum_{i,j=1}^{s}\alpha_i\alpha_j df_i\wedge dg_j + \sum_{k=1}^{m}\sum_{i,j=1}^{s}\alpha_k\beta_j^{(1)}\hat{I}_k df_i\wedge d\gamma_i^k \\ &+ \sum_{k=1}^{m}\sum_{i,j=1}^{s}\alpha_j\beta_i^{(1)}\hat{I}_k d\sigma_i^k\wedge dg_j + \sum_{k,l=1}^{m}\sum_{i,j=1}^{s}\beta_i^{(1)}\beta_j^{(1)}\hat{I}_k\hat{I}_l d\sigma_i^k\wedge d\gamma_i^k. \end{aligned} \tag{A.1}$$

Differentiating (2.7), we obtain

$$\begin{aligned} dp_i^{(0)} &= dP_n + h\sum_{j=1}^{s}A_{ij}^0 df_j + \sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0\hat{I}_l d\sigma_j^l, \\ dq_i^{(0)} &= dQ_n + h\sum_{j=1}^{s}A_{ij}^0 dg_j + \sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^0\hat{I}_l d\gamma_j^l, \\ dp_i^{(k)} &= dP_n + h\sum_{j=1}^{s}A_{ij}^1 df_j + \sum_{j=1}^{s}B_{ij}^1\hat{I}_k d\sigma_j^l + \sum_{l=1,l\neq k}^{m}\sum_{j=1}^{s}B_{ij}^3\hat{I}_l d\sigma_j^l, \\ dq_i^{(k)} &= dQ_n + h\sum_{j=1}^{s}A_{ij}^1 dg_j + \sum_{j=1}^{s}B_{ij}^1\hat{I}_k d\gamma_j^l + \sum_{l=1,l\neq k}^{m}\sum_{j=1}^{s}B_{ij}^3\hat{I}_l d\gamma_j^l. \end{aligned} \tag{A.2}$$

By (A.2), for $i=1,\cdots,s$, we obtain

$$\begin{aligned}
&dP_n\wedge dg_j = dp_i^{(0)}\wedge dg_j - h\sum_{i=1}^{s}A_{ji}^{0}df_i\wedge dg_j - \sum_{l=1}^{m}\sum_{i=1}^{s}B_{ji}^{0}\hat{I}_l d\sigma_i^l\wedge dg_j,\\
&df_i\wedge dQ_n = df_i\wedge dq_i^{(0)} - h\sum_{j=1}^{s}A_{ij}^{0}df_i\wedge dg_j - \sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^{0}\hat{I}_l df_i\wedge d\gamma_j^l,\\
&dP_n\wedge d\gamma_j^k = dp_i^{(k)}\wedge d\gamma_j^k - h\sum_{i=1}^{s}A_{ji}^{1}df_i\wedge d\gamma_j^k - \sum_{i=1}^{s}B_{ji}^{1}\hat{I}_k d\sigma_i^k\wedge d\gamma_j^k - \sum_{l=1}^{m}\sum_{i=1}^{s}B_{ji}^{3}\hat{I}_l d\sigma_i^l\wedge d\gamma_j^k,\\
&d\sigma_i^k\wedge dQ_n = d\sigma_i^k\wedge dq_i^{(k)} - h\sum_{j=1}^{s}A_{ij}^{1}d\sigma_i^k\wedge dg_j - \sum_{j=1}^{s}B_{ij}^{1}\hat{I}_k d\sigma_i^k\wedge d\gamma_j^k - \sum_{l=1}^{m}\sum_{j=1}^{s}B_{ij}^{3}\hat{I}_l d\sigma_k^l\wedge d\gamma_j^l.
\end{aligned}\tag{A.3}$$

Substituting (A.3) into (A.1) yields

$$\begin{aligned}
dP_{n+1}\wedge dQ_{n+1} =& dP_n\wedge dQ_n + h\sum_{i=1}^{s}\alpha_i(dp_i^{(0)}\wedge dg_j + df_j\wedge dq_i^{(0)})\\
&+\sum_{k=1}^{m}\sum_{i=1}^{s}\beta_i^{(1)}\hat{I}_k(dp_i^{(k)}\wedge d\gamma_j^k + d\sigma_i^k\wedge dq_i^{(k)})\\
&-h^2\sum_{i,j=1}^{s}(\alpha_i\alpha_j - \alpha_j A_{ji}^0 - \alpha_i A_{ij}^0)df_i\wedge dg_j\\
&+h\sum_{i,j=1}^{s}\sum_{k=1}^{m}\hat{I}_k(\alpha_i\beta_j - \alpha_i B_{ij}^0 - \beta_j A_{ji}^1)df_i\wedge d\gamma_j^k\\
&+h\sum_{i,j=1}^{s}\sum_{k=1}^{m}\hat{I}_k(\alpha_i\beta_j - \alpha_i B_{ij}^0 - \beta_j A_{ji}^1)d\sigma_i^k\wedge dg_j\\
&+\sum_{i,j=1}^{s}\sum_{k=1}^{m}\hat{I}_k\hat{I}_k(\beta_i\beta_j - \beta_j B_{ji}^1 - \beta_i B_{ij}^1)d\sigma_i^k\wedge d\gamma_j^k\\
&+\sum_{i,j=1}^{s}\sum_{k,l=1}^{m}\hat{I}_k\hat{I}_l(\beta_i\beta_j - \beta_j B_{ji}^3 - \beta_i B_{ij}^3)d\sigma_i^k\wedge d\gamma_j^l.
\end{aligned}\tag{A.4}$$

Consider the second term in the right-hand side of (A.4). We have

$$\begin{aligned}
dp_i^{(0)}\wedge dg_j + df_j\wedge dq_i^{(0)} =& \sum_{k=1}^{d}(dp_i^{(0)k}\wedge dg_j^k + df_j^k\wedge dq_i^{(0)k})\\
=& \sum_{k,l=1}^{d}\Bigg(\frac{\partial g_i^k}{\partial p_l^{(0)}}dp_i^{(0)k}\wedge dp_i^{(0)l} + \frac{\partial g_i^k}{\partial q_l^{(0)}}dp_i^{(0)k}\wedge dq_i^{(0)l}\\
&+\frac{\partial f_i^k}{\partial p_l^{(0)}}dp_i^{(0)l}\wedge dq_i^{(0)k} + \frac{\partial f_i^k}{\partial q_l^{(0)}}dq_i^{(0)l}\wedge dq_i^{(0)k}\Bigg).
\end{aligned}\tag{A.5}$$

Taking into account that the exterior product is skew-symmetric and f and g satisfy condition (2.2), it is not difficult to see that this expression vanishes. In the similar way as in the proof of (A.5), we can deduce that

$$dp_i^{(k)} \wedge d\gamma_j^k + d\sigma_i^k \wedge dq_i^{(k)} = 0. \tag{A.6}$$

Inserting (2.8), (2.9), (2.10), (2.11), (A.5) and (A.6) into (A.1), we see that

$$dP_{n+1} \wedge dQ_{n+1} = dP_n \wedge dQ_n.$$

The proof of the theorem is complete.

Acknowledgments

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