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On Dissipation and Dispersion Errors Optimization, A-Stability and SSP Properties

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Abstract. In a recent paper (Du and Ekaterinaris, 2016) optimization of dissipation and dispersion errors was investigated. A Diagonally Implicit Runge-Kutta (DIRK) scheme was developed by using the relative stability concept, i.e. the ratio of absolute numerical stability function to analytical one. They indicated that their new scheme has many similarities to one of the optimized Strong Stability Preserving (SSP) schemes. They concluded that, for steady state simulations, time integration schemes should have high dissipation and low dispersion. In this note, dissipation and dispersion errors for DIRK schemes are studied further. It is shown that relative stability is not an appropriate criterion for numerical stability analyses. Moreover, within absolute stability analysis, it is shown that there are two important concerns, accuracy and stability limits. It is proved that both A-stability and SSP properties aim at minimizing the dissipation and dispersion errors. While A-stability property attempts to increase the stability limit for large time step sizes and by bounding the error propagations via minimizing the numerical dispersion relation, SSP optimized method aims at increasing the accuracy limits by minimizing the difference between analytical and numerical dispersion relations. Hence, it can be concluded that A-stability property is necessary for calculations under large time-step sizes and, more specifically, for calculation of high diffusion terms. Furthermore, it is shown that the oscillatory behavior, reported by Du and Ekaterinaris (2016), is due to Newton method and the tolerances they set and it is not related to the employed temporal schemes.

AMS subject classifications: 65N12

Key words: Diagonally Implicit Runge-Kutta methods, dissipation and dispersion, optimization, numerical stability, steady state.

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1 Introduction

Steady state solutions could be sought as the long-time mean solutions of the unsteady problems (Du and Ekaterinaris, 2016). However, dissipation and dispersion errors are the main obstacles to achieve long time-step sizes and accordingly to decrease the calculation time.

Among different temporal integrator schemes, Runge-Kutta methods have attracted attentions, as they are single-step methods and have free parameters which could lead to optimization of dissipation and dispersion errors. Excessive research in the literature has been made to control and bound dissipation and dispersion errors to increase the range of stability and accuracy. Consequently, numerous stability properties have been introduced including A-stability property. The reader is referred to ODEs books for more details (e.g. Hairer and Wanner, 1996).

The Strong Stability Preserving (SSP) Runge-Kutta methods are well-known due to their non-oscillatory behavior in shock and discontinuity problems. These methods were designed as convex combinations of Forward Euler (FE) method within limited radius of absolute monotonicity. This class of methods was further developed by Gottlieb and Shu (1998). Then, Ketcheson (2009) developed optimal implicit SSP RungeKutta methods up to order six with eleven stages.

The objective of this paper is to further investigate the stability analysis within studying dissipation and dispersion errors, in order to discuss the conclusions of Du and Ekaterinaris (2016) and to discuss their proposed DIRK scheme.

Du and Ekaterinaris (2016) indicated that this new scheme has many similarities with the three-stage fourth order SSP optimized DIRK scheme. They also described their proposed scheme, so called DIRK-D, as a more accurate model for low wavenumber components than other schemes they employed. However, as will be discussed, in stability analyses of temporal schemes, the main attention is on time step sizes. The wavenumber is assumed as a fixed variable, which basically would be the highest one. It will be shown that the source of instability, imposed by grid mesh, is due to high wavenumbers.

Relative stability analysis, i.e. the ratio of absolute numerical stability function to analytical one, was introduced by Hairer and Wanner (1996) within the concept of Order Star. Du and Ekaterinaris (2016) used relative stability function to design the optimized three-stage fourth order DIRK scheme, DIRK-D, and to examine its performances. However, the Order Star is mainly useful in proving relation between stability and achievable order of accuracy and this idea is not useful for judging the stability. Indeed, absolute stability is the more practical one (Leveque, 2007).

In Section 2, the relative stability analysis is studied further in order to show that this concept is not useful for optimization of the dissipation and dispersion errors. Meanwhile, as indicated by Leveque (2007) and shown in the present paper, this concept just shows the order of accuracy and truncation error. Du and Ekaterinaris (2016) indicated that, for advection-diffusion system, the amplification factor needs to include contribution of physical and numerical diffusion. In contrast, it will be shown that this contri-