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A Gradient-Enhanced ℓ_1 Approach for the Recovery of Sparse Trigonometric Polynomials

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Abstract. In this paper, we discuss a gradient-enhanced ℓ_1 approach for the recovery of sparse Fourier expansions. By *gradient-enhanced* approaches we mean that the directional derivatives along given vectors are utilized to improve the sparse approximations. We first consider the case where both the function values and the directional derivatives at sampling points are known. We show that, under some mild conditions, the inclusion of the derivatives information can indeed decrease the coherence of measurement matrix, and thus leads to the improved the sparse recovery conditions of the ℓ_1 minimization. We also consider the case where either the function values or the directional derivatives are known at the sampling points, in which we present a sufficient condition under which the measurement matrix satisfies RIP, provided that the samples are distributed according to the uniform measure. This result shows that the derivatives information plays a similar role as that of the function values. Several numerical examples are presented to support the theoretical statements. Potential applications to function (Hermite-type) interpolations and uncertainty quantification are also discussed.

AMS subject classifications: 65D15, 41A10, 41A63

Key words: Gradient-enhanced ℓ_1 minimization, compressed sensing, sparse Fourier expansions, restricted isometry property, mutual incoherence.

1 Introduction

Compressed sensing (CS), introduced by Candes, Romberg & Tao [14] and Donoho [25], has been a hot research field in recent years [2,3,7,12,16–19,29,37]. The main motivation for CS is that many real world signals can be well approximated by sparse ones, more precisely, they can be approximated by an expansion in terms of a suitable basis (e.g,

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Fourier basis), which has only a few non-vanishing terms. CS predicts that sparse vectors in high dimensions can be recovered from few measurements. Recently, CS has been successfully applied in many areas, such as imaging [42,46], radar [24], wireless communication [36], Magnetic Resonance Imaging [15], uncertainty quantification [23,26,32], to name a few.

In this work, we consider the recovery of sparse Fourier expansions by ℓ_1 minimization. Particularly, we propose in this work a *gradient-enhanced* ℓ_1 minimization, which means that the *gradient information* of the Fourier expansions at samples are included in the ℓ_1 minimization. In other words, we consider the ℓ_1 minimization with both function evaluations and the derivative information and we will study the effect of the derivative on the performance of ℓ_1 minimization. We next briefly introduce the motivation for considering the derivative information:

- **Uncertainty quantification (UQ) for random PDEs.** In UQ, one represents the solution as a linear combination of certain bases, such as orthonormal polynomials and Fourier bases, and then uses a few sample evaluations to obtain the sparse approximation of the solutions. Given the sample evaluations, the derivative information can sometime be obtained in a cheaper way, e.g, by solving the adjoint equations [13, 21, 41]. This is a well known approach in the numerical PDE community, and has been used in UQ studies [4,25,28,35]. Naturally, one hopes to use both the function evaluations and the gradient information to enhance the sparse approximation of the solution to PDE.

- **Sparse Hermite-type interpolation.** Recently, one employs the result in CS to study the sparse interpolation [1,14,40,52] which shows many potential applications. To find a function from a finite dimensional function space which interpolates function value and derivatives at some points is called Hermite interpolation [5,31,44,48]). The gradient-enhanced approach here can be viewed as a sparse Hermite-type interpolation which seeks to find a *sparse* interpolation from the function values and derivatives. Hence, the results in this paper also show the connection between compressed sensing and the classical approximation theory.

A simple observation is that the gradient-enhanced ℓ_1 approach uses more information than the standard approach (or in other words, enhance the row size of the sensing matrix), and this opens up the possibility to improve the sparse recovery ability of the ℓ_1 minimization. This work aims at analyzing the *gradient-enhanced* ℓ_1 minimization and provide precise conditions under which the new approach can indeed improve the sparse recovery. To this end, we begin with some preliminaries on CS.

1.1 Compressed sensing

The aim of compressed sensing is to find a sparse solution to linear equations

$$\mathbf{\Phi c} = \mathbf{f},$$