Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0190

Vol. **22**, No. 3, pp. 599-619
September 2017

Interaction of a Vortex Induced by a Rotating Cylinder with a Plane

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Received 2 November 2016; Accepted (in revised version) 7 February 2017

Abstract. In this article,we study theoretically and numerically the interaction of a vortex induced by a rotating cylinder with a perpendicular plane. We show the existence of weak solutions to the swirling vortex models by using the Hopf extension method, and by an elegant contradiction argument, respectively. We demonstrate numerically that the model could produce phenomena of swirling vortex including boundary layer pumping and two-celled vortex that are observed in potential line vortex interacting with a plane and in a tornado.

AMS subject classifications: 76U05, 76D03, 76D17, 65Z05

Key words: Tornado, swirling vortex, Navier-Stokes, axisymmetric flow, boundary layer.

1 Introduction

The aim of this article is to study the interaction of a vertical line vortex with a horizontal plane as a first simplified model for tornadoes. The study is partly theoretical and partly numerical. In the analytical part, the line vortex is replaced by a vertical rotating cylinder of small radius σ, and we show the existence of an axisymmetric weak solution to the stationary Navier-Stokes equations, when $\sigma > 0$. In the numerical part, we examine carefully the fluid flow near the plane boundary and around the cylinder.

The study of a line vortex interacting with a plane perpendicular to the vortex core, even in the simple axisymmetric setting, is important, as the structures of the resulting swirling vortex (sometimes exact solutions to the stationary Navier-Stokes equations) can give insight into the dramatic phenomenon of a tornado. Goldshtik [9] discovered a family of conical self-similar swirling vortex solutions to the axisymmetric Navier-Stokes equations resulting from a vertical potential line vortex of constant circulation interacting

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with an infinite orthogonal plane. However, Goldshtik's vortex solutions exist only for small Reynolds numbers ($Re<Re^*=5.53$), since the solutions are assumed to be bounded at the vortex axis [8]. Serrin partially resolved Goldshtik's paradox and showed the existence of self-similar vortex solutions for all Reynolds numbers, if one allows singularity formation at the vortex core [24]. Due to the lack of boundary conditions, Serrin's solutions depend on an additional parameter that needs to be specified as an input to the physical system. The parameter amounts to specifying a boundary condition for the pressure on the plane surface. Nevertheless, Serrin shows that these solutions have rich structures including two-celled vortex. Recently, there has been research on modifying the scaling of the velocity/radial distance dependence in Serrin's vortex solutions, based on radar data observation [4].

The major controversy in Serrin's idealized model of a line vortex stems from the vortex singularity at the vortex axis which serves as a source of momentum. In the present work, we regularize the line vortex by a rotating cylinder of small radius. Serrin's model of a line vortex can be viewed as an asymptotic limit of the rotating cylinder when its radius approaches zero. Moreover, as the vortex singularity is regularized, no additional physical parameters are needed in our model other than the circulation and kinematic viscosity.

Now we describe the problem of tornado-like vortex driven by a uniform rotating cylinder of a small radius in detail. The natural coordinates system for this problem is the cylindrical coordinates (r,θ,z). Let (u,v,w) be the velocity vector in the cylindrical coordinates. Then the axisymmetric flow is governed by the following dimensional steady-state Navier-Stokes equations

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\Big(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\Big), \tag{1.1}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\Big(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\Big), \tag{1.2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\Big(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\Big), \tag{1.3}$$

$$\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}=0, \tag{1.4}$$

where ρ is the density, and ν is the kinematic viscosity. The domain is defined as $\Omega=\{(r,\theta,z)\mid r\geq\sigma,\theta\in(0,2\pi),0\leq z\leq L\}$. We decompose the boundary $\partial\Omega$ into several parts such that $\partial\Omega=\Gamma_i\cup\Gamma_l\cup\Gamma_u$ with $\Gamma_i=\{r\geq\sigma,z=0\}$, $\Gamma_l=\{r=\sigma,0\leq z\leq L\}$ and $\Gamma_u=\{r\geq\sigma,z=L\}$. The boundary conditions for the flow that we would like to impose are

$$(u,v,w)|_{\Gamma_i}=0,\quad (u,v,w)|_{\Gamma_l}=\Big(0,\frac{\gamma}{2\pi\sigma},0\Big),\quad \Big(\frac{\partial u}{\partial z},\frac{\partial v}{\partial z},w\Big)|_{\Gamma_u}=0. \tag{1.5}$$

Here γ is the circulation prescribed on the surface of the cylinder. Recall that a potential vortex line is given by $u=w=0$, $v=\frac{\gamma}{2\pi r}=-\phi'(r)$ with the potential $\phi(r)=-\frac{\gamma}{2\pi}\ln r$. The