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Mesh Density Functions Based on Local Bandwidth Applied to Moving Mesh Methods

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Abstract. Moving mesh methods provide an efficient way of solving partial differential equations for which large, localised variations in the solution necessitate locally dense spatial meshes. In one-dimension, meshes are typically specified using the arclength mesh density function. This choice is well-justified for piecewise polynomial interpolants, but it is only justified for spectral methods when model solutions include localised steep gradients. In this paper, one-dimensional mesh density functions are presented which are based on a spatially localised measure of the bandwidth of the approximated model solution. In considering bandwidth, these mesh density functions are well-justified for spectral methods, but are not strictly tied to the error properties of any particular spatial interpolant, and are hence widely applicable. The bandwidth mesh density functions are illustrated in two ways. First, by applying them to Chebyshev polynomial approximation of two test functions, and second, through use in periodic spectral and finite-difference moving mesh methods applied to a number of model problems in acoustics. These problems include a heterogeneous advection equation, the viscous Burgers' equation, and the Korteweg-de Vries equation. Simulation results demonstrate solution convergence rates that are up to an order of magnitude faster using the bandwidth mesh density functions than uniform meshes, and around three times faster than those using the arclength mesh density function.

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1 Introduction

Many scientific and engineering problems require solutions to partial differential equations (PDEs). When smooth, these solutions can be efficiently computed using spectral methods. However, often solutions are not equally smooth everywhere. In particular, they might exhibit features which are tightly localised in space. These include

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shock fronts, narrow pulses, and sharp corners. Such features require dense computational meshes to accurately resolve. Because spectral methods typically use standardised meshes, the global mesh density is determined by the sampling requirements of these localised features. This results in much of the spatial domain being oversampled, increasing computational expense for no accuracy gain. As an example of when this can become a critical issue, three-dimensional, full-wave simulations of nonlinear medical ultrasound fields may require many tens of gigabytes of memory to store acoustic field variables at each time-step due to large, densely sampled simulation domains [16]. These sampling requirements arise when acoustic nonlinearity causes very high frequencies to form, often within small regions where the acoustic pressure is particularly high.

Adaptive moving mesh methods can reduce the trade-off between accuracy and computational expense by providing more optimal sampling. They place mesh nodes according to a monitor function (sometimes called a mesh density function in one dimension) that is computed from (and locally dependent on) the calculated solution itself. Moving mesh methods have traditionally been implemented using finite-difference and finite-element methods, but spectral implementations offer the opportunity to improve computational efficiency further. Some examples of spectral moving mesh methods include Fourier [8,9], Galerkin [20], and Chebyshev [22] types. These all used the arclength monitor function, which clusters mesh nodes according to the gradient of the model solution. For these problems, this choice is justified by physical considerations: the model solutions in all cases feature localised steep gradients. However, it is not clear why the arclength monitor function might produce a mesh that is optimal.

One justification for the arclength monitor function is given in [15, §2.4]. Here, it is shown that derivative-based monitor functions can be derived from interpolation error bounds for piecewise polynomial interpolants. The arclength monitor function, while not strictly optimal, can be seen to be very similar to these. But this approach does not naturally extend to spectral interpolants. An alternative is to directly consider smoothness properties of the approximated solution itself. A notable one-dimensional example is found in the work of Tee et al. [11, 12, 23, 24]. This approach is designed for solutions whose analytic continuations contain singularities. It works by first approximating the analytic continuation, after which a mesh mapping is computed that is parametrised by the locations of the approximated singularities. These mesh mappings seek to ensure that a spectral interpolant through the composition of the approximated solution and inverse mesh mapping converges on the true solution faster than a spectral interpolant through the solution alone. In [25], a mesh density function is presented which is based on Tee et al.'s approach. This work demonstrated that a singularity-based mesh density function was significantly more effective than the arclength mesh density function in reducing the trade-off between accuracy and computational expense. However, an obvious limitation of this approach is that it requires the model solution's analytic continuation to include singularities (or at least near-singular behaviour).

Recently, Subich [21] demonstrated a more general one-dimensional spectral moving mesh method, which uses a mesh density function that is given by the envelope of the