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# Computing Optimal Forcing Using Laplace Preconditioning

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**Abstract.** For problems governed by a non-normal operator, the leading eigenvalue of the operator is of limited interest and a more relevant measure of the stability is obtained by considering the harmonic forcing causing the largest system response. Various methods for determining this so-called optimal forcing exist, but they all suffer from great computational expense and are hence not practical for large-scale problems. In the present paper a new method is presented, which is applicable to problems of arbitrary size. The method does not rely on timestepping, but on the solution of linear systems, in which the inverse Laplacian acts as a preconditioner. By formulating the search for the optimal forcing as an eigenvalue problem based on the resolvent operator, repeated system solves amount to power iterations, in which the dominant eigenvalue is seen to correspond to the energy amplification in a system for a given frequency, and the eigenfunction to the corresponding forcing function. Implementation of the method requires only minor modifications of an existing timestepping code, and is applicable to any partial differential equation containing the Laplacian, such as the Navier-Stokes equations. We discuss the method, first, in the context of the linear Ginzburg-Landau equation and then, the two-dimensional lid-driven cavity flow governed by the Navier-Stokes equations. Most importantly, we demonstrate that for the lid-driven cavity, the optimal forcing can be computed using a factor of up to 500 times fewer operator evaluations than the standard method based on exponential timestepping.

**AMS subject classifications**: 76M25, 65F08, 76E15, 76E09, 65F15

**Key words**: Hydrodynamic stability, optimal forcing, Laplace preconditioner, iterative methods, eigenvalue problem, Ginzburg-Landau equation, lid-driven cavity flow.

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## 1 Introduction

In hydrodynamic stability, a classical analysis generally consists of two parts — the determination of a basic state about which the governing equations may be linearized, and the calculation of eigenvalues of the Jacobian $\mathcal{A}$. For non-normal operators, other considerations may be more relevant. For example, solutions may experience transient growth even when all of the eigenvalues are located in the left half of the complex plane, and in a non-linear framework trigger subcritical transition [10,30,31]. Another type of analysis concerns the amplification due to a harmonic driving force $f(x)e^{i\omega t}$, where we seek to determine the temporal frequency $\omega$ and spatial profile $f$ that cause the largest energy amplification in the system.

The purpose of this paper is to introduce a novel iterative matrix-free method for computing the optimal forcing of a driven system. This method is best explained by placing it in the context of those used to carry out linear stability analysis, so we begin by surveying these techniques. Denoting by $\mathcal{A}$ the governing operator linearized about a basic state, perturbations $q(x,t)$ obey

$$\frac{\partial q}{\partial t} = \mathcal{A}q. \tag{1.1}$$

The governing operator $\mathcal{A}$ is considered to be spatially dependent, either via the geometrical specifications of the problem, or through a spatially-dependent basic state about which the evolution equations have been linearized, or both. Perturbations $q$ may depend on one, two, or three spatial dimensions. If there is only one spatial dimension, the governing operator can be formulated and treated explicitly. For higher-dimensional systems, if one or two of the spatial directions are homogeneous, then the eigenfunctions are trigonometric or exponential in those directions and the linearized operator is banded or block-diagonal [12]. In such cases, it may still be possible to determine the eigenvalues and eigenfunctions (denoted by eigenpairs) of $\mathcal{A}$ through direct methods.

With increased geometrical complexity, an explicit representation and a full diagonalization of the operator are usually too costly in terms of storage and computational power and it becomes necessary to use matrix-free methods to find the desired eigenpairs. A timestepping algorithm for solving (1.1), which carries out the action of an approximation to the exponential operator $\exp(\mathcal{A}\Delta t)$, is a natural means for doing so. Integrating the linearized equations (1.1) in time is equivalent to carrying out the power method on $\exp(\mathcal{A}\Delta t)$, and will converge to the leading eigenfunction.

Turning to the topic of this paper, when a system is linearly stable, it may nevertheless undergo amplification due to a harmonic driving force, as described by

$$\frac{\partial q}{\partial t} = \mathcal{A}q + f e^{i\omega t}. \tag{1.2}$$

If all of the eigenvalues of $\mathcal{A}$ have negative real part, then $q(x,t) \to -(\mathcal{A} - i\omega\mathcal{I})^{-1} f(x) e^{i\omega t}$