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# An Explicit Formula for Two-Dimensional Singly-Periodic Regularized Stokeslets Flow Bounded by a Plane Wall

Forest O. Mannan[1,*] and Ricardo Cortez[1]

[1] *Department of Mathematics, Tulane University, New Orleans, Louisiana 70118-5665, USA.*

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**Abstract.** We derive a closed form expression for the regularized Stokeslet in two space dimensions with periodic boundary conditions in the $x$-direction and a solid plane wall at $y=0$. To accommodate the no-slip condition on the wall, a system of images for the regularized Stokeslets was used. The periodicity is enforced by writing all elements of the image system in terms of a Green's function whose periodic extension is known. Although the formulation is derived in the context of regularized Stokeslets, the expression for the traditional (singular) Stokeslet is easily found by taking the limit as the regularization parameter approaches zero. The new formulation is validated by comparing results of two test problems: the Taylor infinite waving sheet and the motion of a cylinder moving near a wall. As an example of an application, we use our formulation to compute the motion and flow generated by cilia using a model that does not prescribe the motion so that the beat period and synchronization of neighboring cilia are a result of the forces developed along the cilia.

**AMS subject classifications**: 76D07, 76Z05

**Key words**: Regularized Stokeslets, singly-periodic flow, method of images.

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## 1 Introduction

Spatially periodic systems have been used for many years to study phenomena with large numbers of particles or with repeated structures immersed in a viscous fluid. Examples are the dynamics of sedimenting particles [14, 36], sedimentation of fibers [33], instabilities associated with some flows with periodicity [12] and with active suspensions [34], flow past arrays of spheres [18, 43], and flows generated by arrays of cilia [42].

*Corresponding author. *Email addresses:* `fmannan@tulane.edu` (F. O. Mannan), `rcortez@tulane.edu` (R. Cortez)

 

In cases of large numbers of particles in space, it becomes computationally prohibitive to account for each particle, so models with a smaller number of the particles in a periodic domain are used to capture the behavior of those systems. In most of these cases, the models begin with an infinite sum representing the contributions of the periodic replication of forces and proceed to truncate the sums based on rapidly converging formulations [4, 9, 18, 24, 25, 31, 40]. Since many of these flows can occur near a wall, as is the case of cilia attached to a surface, the swimming of sperm, the sedimentation of particles, and flow past obstacles [16, 22, 35], there has been a need to develop methods to satisfy a no-flow condition at an infinite plane [1, 3] in otherwise unbounded domains.

There is continued interest in modeling two-dimensional Stokes flow in unbounded domains [11, 17] or with periodic boundary conditions either in both directions or only in one direction (see e.g. [6, 27, 31, 34]). Marple et al. [27] recently considered the flow of vesicles in a channel by computing two dimensional Stokes flow with periodic boundary conditions at the channel outlet. Their approach is to use direct free-space summation that includes the two nearest periodic images and approximate the contributions from the more distant images using proxy sources on the boundary of an auxiliary domain. Bryngelson and Freund [6] used boundary integral methods to simulate two-dimensional channel flow with a dense suspension of deforming fluid-filled capsules. Crowdy and Or [11] developed a two-dimensional model to compute the motion of a swimmer near a wall using complex analysis. Importantly, their results show sufficient agreement with experiments and three-dimensional models to conclude that 2D models can provide insight into the dynamics of low-Reynolds number swimmers.

Here we describe an efficient method for two-dimensional flows generated by cilia or particle suspensions near a plane wall at $y=$ constant and with periodic boundary conditions in the $x$-direction (see Fig. 1). Our motivation comes from the motion of cilia, which are attached to a surface and beat to generate coordinated flows. The effect of multiple cilia beating next to each other can be modeled by imposing periodicity while the surface lined by the cilia is modeled as an infinite wall where no-flow boundary conditions are satisfied. We proceed by first adapting the method of regularized Stokeslets [8] for the case of periodic boundary conditions in the $x$-direction through the use of a singly-periodic Green's function. We then derive the system of images [1, 10] that analytically cancels the flow at the wall. The result is an explicit formula for the flow due to regularized singly-periodic Stokeslets near a plane wall in two dimensions, which provides a straight forward way to investigate the flows of interest.

There have been many approaches to simulating cilia but they generally fall into two categories. Either the shape of the cilium beat is assumed to be known based on observed biological data [20, 23] or the motion of the cilium emerges from the model of an internal force generating mechanism [7, 15, 16, 28]. While modeling the internal mechanisms is more challenging, it has the feature that the motion of the cilium itself is coupled to the fluid and is a priori unknown. With this approach, properties that depend on hydrodynamic coupling can be investigated, including how the beat frequency depends on the spacing between cilia and the synchronization of cilia beat patterns. In the present appli-