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Dispersive Shallow Water Wave Modelling. Part I: Model Derivation on a Globally Flat Space

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Abstract. In this paper we review the history and current state-of-the-art in modelling of long nonlinear dispersive waves. For the sake of conciseness of this review we omit the unidirectional models and focus especially on some classical and improved BOUSSINESQ-type and SERRE–GREEN–NAGHDI equations. Finally, we propose also a unified modelling framework which incorporates several well-known and some less known dispersive wave models. The present manuscript is the first part of a series of two papers. The second part will be devoted to the numerical discretization of a practically important model on moving adaptive grids.

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1 Introduction

The history of nonlinear dispersive modelling goes back to the end of the XIX$^{\text{th}}$ century [22]. At that time J. BOUSSINESQ (1877) [14] proposed (in a footnote on page 360) the celebrated KORTEWEG–DE VRIES equation, re-derived later by D. KORTEWEG & G. DE VRIES (1895) [48]. Of course, J. BOUSSINESQ proposed also the first BOUSSINESQ-type

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equation [12,13] as a theoretical explanation of *solitary waves* observed earlier by J. RUSSELL (1845) [67]. After this initial active period there was a break in this field until 1950's. The silence was interrupted by the new generation of 'pioneers' — F. SERRE (1953) [69,70], C.C. MEI & LE MÉHAUTÉ (1966) [57] and D. PEREGRINE (1967) [66] who derived modern nonlinear dispersive wave models. After this time the modern period started, which can be characterized by the proliferation of journal publications and it is much more difficult to keep track of these records. Subsequent developments can be conventionally divided in two classes:

1. Application and critical analysis of existing models in new (and often more complex) situations;
2. Development of new high-fidelity physical approximate models.

Sometimes both points can be improved in the same publication. We would like to mention that according to our knowledge the first applications of PEREGRINE's model [66] to three-dimensional practical problems were reported in [1,68].

In parallel, scalar model equations have been developed. They describe the unidirectional wave propagation [31,64]. For instance, after the above-mentioned KdV equation, its regularized version was proposed first by PEREGRINE (1966) [65], then by BENJAMIN, BONA & MAHONY (1972) [6]. Now this equation is referred to as the Regularized Long Wave (RLW) or BENJAMIN–BONA–MAHONY (BBM) equation. In [6] the well-posedness of RLW/BBM equation in the sense of J. HADAMARD was proven as well. Even earlier WHITHAM (1967) [76] proposed a model equation which possesses the dispersion relation of the full EULER equations (it was constructed in an ad-hoc manner to possess this property). It turned out to be an excellent approximation to the EULER equations in certain regimes [60]. Between unidirectional and bi-directional models there is an intermediate level of scalar equations with second order derivatives in time. Such an intermediate model was proposed, for example, in [47]. Historically, the first BOUSSINESQ-type equation proposed by J. BOUSSINESQ [14] was in this form as well. The main advantage of these models is their simplicity on one hand, and the ability of providing good quantitative predictions on the other hand.

One possible classification of existing nonlinear dispersive wave models can be made upon the choice of the horizontal velocity variable. Two popular choices were suggested in [66]. Namely, one can use the depth-averaged velocity variable (see *e.g.* [24,33,36,68,77,78]). Usually, such models enjoy nice mathematical properties such as the exact mass conservation equation. The second choice consists in taking the trace of the velocity on a surface defined in the fluid bulk $y = \mathcal{Y}(x,t)$. Notice, that surface $\mathcal{Y}(x,t)$ may eventually coincide with the free surface [21] or with the bottom [2,57]. This technique was used for the derivation of several Boussinesq type systems with flat bottom, initially in [11] and later in [8,9] and analysed thoroughly theoretically and numerically in [3–5,8,10,25]. Sometime the choice of the surface is made in order to obtain a model with improved dispersion characteristics [11,56,75]. One of the most popular model of this class is due to O. NWOGU (1993) [63] who proposed to use the horizontal velocity defined at $y =$