Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0179b

Vol. **23**, No. 1, pp. 30-92
January 2018

Dispersive Shallow Water Wave Modelling. Part II: Numerical Simulation on a Globally Flat Space

Gayaz Khakimzyanov[1], Denys Dutykh[2,*], Oleg Gusev[1] and Nina Yu. Shokina[1]

[1] *Institute of Computational Technologies, Siberian Branch of the Russian Academy of Sciences, Novosibirsk 630090, Russia.*
[2] *LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc, Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France.*

Received 23 October 2016; Accepted (in revised version) 26 June 2017

Abstract. In this paper we describe a numerical method to solve numerically the weakly dispersive fully nonlinear SERRE–GREEN–NAGHDI (SGN) celebrated model. Namely, our scheme is based on reliable finite volume methods, proven to be very efficient for the hyperbolic part of equations. The particularity of our study is that we develop an adaptive numerical model using moving grids. Moreover, we use a special form of the SGN equations where non-hydrostatic part of pressure is found by solving a linear elliptic equation. Moreover, this form of governing equations allows to determine the natural form of boundary conditions to obtain a well-posed (numerical) problem.

AMS subject classifications: 76B15, 76M12, 65N08, 65N06

PACS: 47.35.Bb, 47.35.Fg

Key words: Nonlinear dispersive waves, non-hydrostatic pressure, moving adaptive grids, finite volumes, conservative finite differences.

1 Introduction

In 1967 D. PEREGRINE derived the first two-dimensional BOUSSINESQ-type system of equations [117]. This model described the propagation of long weakly nonlinear waves over a general non-flat bottom. From this landmark study the modern era of long wave modelling started. On one hand researchers focused on the development of new models and in parallel the numerical algorithms have been developed. We refer to [20] for a recent 'reasoned' review of this topic.

*Corresponding author. *Email addresses:* Khak@ict.nsc.ru (G. Khakimzyanov), Denys.Dutykh@univ-savoie.fr (D. Dutykh), gusev_oleg_igor@mail.ru (O. Gusev), Nina.Shokina@googlemail.com (N. Yu. Shokina)

The present manuscript is the continuation of our series of papers devoted to the long wave modelling. In the first part of this series we derived the so-called base model [92], which encompasses a number of previously known models (but, of course, not all of nonlinear dispersive systems). The governing equations of the base model are

$$\mathcal{H}_t + \boldsymbol{\nabla}\cdot[\mathcal{H}\boldsymbol{U}] = 0, \tag{1.1}$$

$$\bar{\boldsymbol{u}}_t + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{u}} + \frac{\boldsymbol{\nabla}\mathscr{P}}{\mathcal{H}} = \frac{\check{p}}{\mathcal{H}}\boldsymbol{\nabla}h - \frac{1}{\mathcal{H}}\Big[(\mathcal{H}\mathscr{U})_t + (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})(\mathcal{H}\mathscr{U}) + \mathcal{H}(\mathscr{U}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{u}} + \mathcal{H}\mathscr{U}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}\Big], \tag{1.2}$$

where $\boldsymbol{U} \stackrel{\text{def}}{:=} \bar{\boldsymbol{u}} + \mathscr{U}$ is the modified horizontal velocity and $\mathscr{U} = \mathscr{U}(\mathcal{H},\bar{\boldsymbol{u}})$ is the closure relation to be specified later. Depending on the choice of this variable various models can be obtained (see [92, Section §2.4]). Variables $\mathscr{P}$ and $\check{p}$ are related to the fluid pressure. The physical meaning of these variables is reminded below in Section 2. In the present paper we propose an adaptive numerical discretization for a particular, but very popular nowadays model which can be obtained from the base model (1.1), (1.2). Namely, if we choose $\mathscr{U} \equiv \boldsymbol{0}$ (thus, $\boldsymbol{U}$ becomes the depth-averaged velocity $\boldsymbol{u}$) then we obtain equations equivalent to the celebrated SERRE–GREEN–NAGHDI (SGN) equations [72,126,127] (rediscovered later independently by many other researchers). This system will be the main topic of our numerical study. Most often, adaptive techniques for dispersive wave equations involve the so-called Adaptive Mesh Refinement (AMR) [121] (see also [15] for nonlinear shallow water equations). The particularity of our study is that we conserve the total number of grid points and the adaptivity is achieved by judiciously redistributing them in space [83, 84]. The ideas of redistributing grid nodes is stemming from the works of BAKHVALOV [7], IL'IN [85] and others [1, 134].

The base model (1.1), (1.2) admits an elegant conservative form [92]:

$$\mathcal{H}_t + \boldsymbol{\nabla}\cdot[\mathcal{H}\boldsymbol{U}] = 0, \tag{1.3}$$

$$(\mathcal{H}\boldsymbol{U})_t + \boldsymbol{\nabla}\cdot\Big[\mathcal{H}\bar{\boldsymbol{u}}\otimes\boldsymbol{U} + \mathscr{P}(\mathcal{H},\bar{\boldsymbol{u}})\cdot\mathbb{I} + \mathcal{H}\mathscr{U}\otimes\bar{\boldsymbol{u}}\Big] = \check{p}\boldsymbol{\nabla}h, \tag{1.4}$$

where $\mathbb{I} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is the identity matrix and the operator $\otimes$ denotes the tensorial product. We note that the pressure function $\mathscr{P}(\mathcal{H},\bar{\boldsymbol{u}})$ incorporates the familiar hydrostatic pressure part $\dfrac{g\mathcal{H}^2}{2}$ well-known from the Nonlinear Shallow Water Equations (NSWE) [11, 43]. By setting $\mathscr{U} \equiv \boldsymbol{0}$ we obtain readily from (1.3), (1.4) the conservative form of the SGN equations (one can notice that the mass conservation equation (1.1) was already in conservative form).

Nonlinear dispersive wave equations represent certain numerical difficulties since they involve mixed derivatives (usually of the horizontal velocity variable, but sometimes of the total water depth as well) in space and time. These derivatives have to be approximated numerically, thus leaving a lot of room for the creativity. Most often the so-called *Method Of Lines* (MOL) is employed [97,120,123,128], where the spatial derivatives